Relationships between Nodes, Connections, and Faces in Planar Graphs

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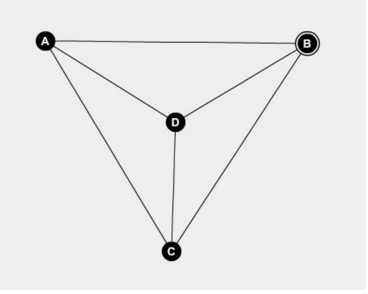
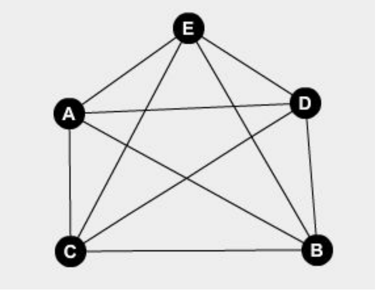
Introduction

A graph is a group of points (also called *nodes*) joined together by lines. There can be curved or straight lines. In order for a graph to be planar, there must be some arrangement of the graph so that the lines only intersect at the nodes.



Planar Graph



Non-Planar Graph

Problem Statement

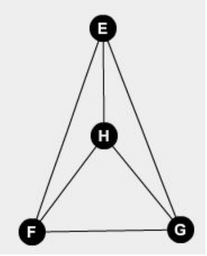
Is there a relationship between the number of possible connections, number of faces, and number of nodes in a planar graph and does the arrangement of the nodes matter? Why?

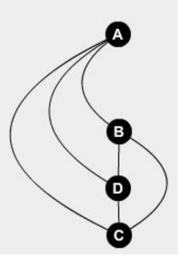
Results

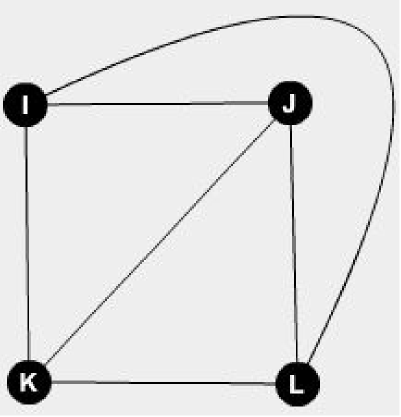
To collect data for this project, I created planar and non-planar graphs and counted the number of connections, faces and nodes.

# of points	# of faces	# of connections
3	2	3
4	4	6
5	6	9
6	8	12
7	10	15
8	12	18
9	14	21
10	16	24
11	18	27
12	20	30
13	22	33
14	24	36

To find these results I created three general shapes of a planar graph to use for each number of nodes:



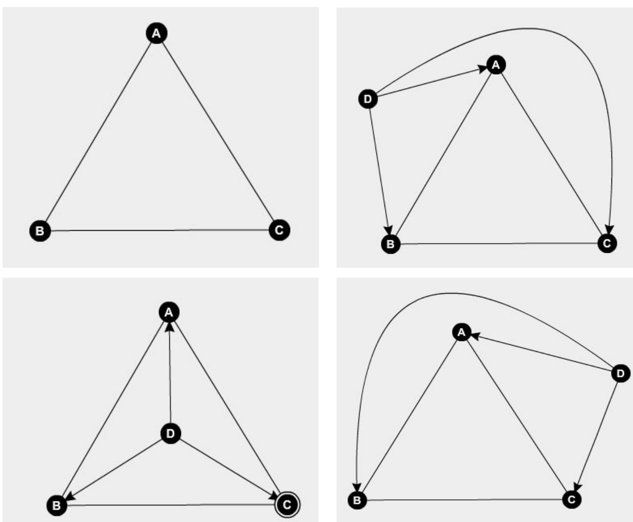




From this data, I found that the relationship between the amount of connections, C, and the amount of nodes, n, was $C = 3(n-2)$.

Once I found this information, I wanted to explain why the equation $C = 3(n-2)$ is multiplying by three, making the amount of connections go up by three every time a node is added.

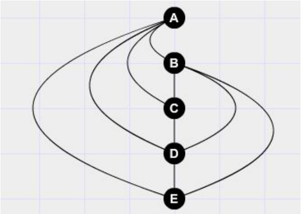
Examples of adding nodes : (Node D is added to triangle ABC.



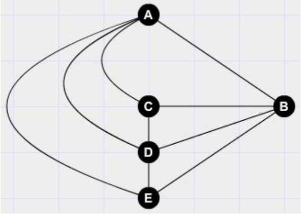
(The lines with arrows at the end are the newly formed connections.)
Top Left : Original triangle ABC.
Top Right: Node D added to the right of Node A.
Bottom Left: Node D is added in the center of triangle ABC.
Bottom Right: Node D is added to the left of Node A.

Why the arrangement of nodes does not matter:

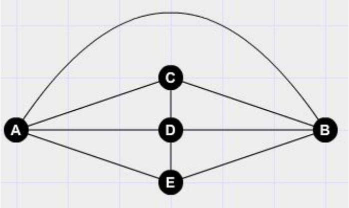
The graphs below have 5 nodes with 9 connections.



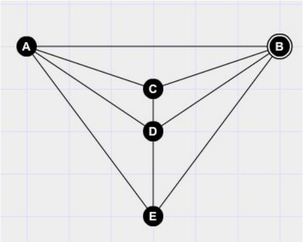
1. This is one kind of planar graph arrangement for 5 nodes and 9 connections.



2. When we move NODE B out to the right, all of it's connections become straight.



3. Now we move NODE A out to the left, creating eight straight lines and one curved line, from NODE A to NODE B.

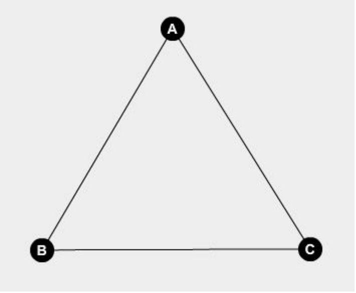


4. Now I have moved both NODE A and NODE B upwards to create a new arrangement made of only straight lines.

Why can an additional node only connect to 3 other nodes?

Picture #1

Here is a triangle ABC.

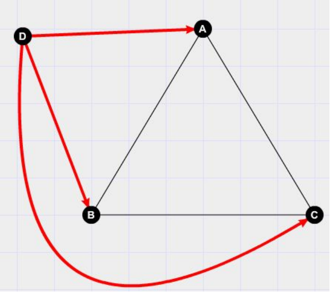


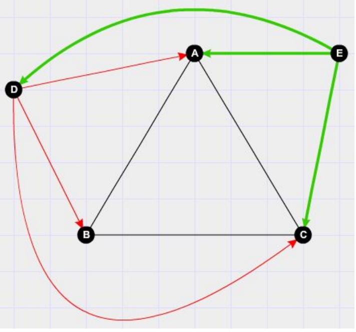
Picture #2

Now we add NODE D.

It connects to NODES A,B,C.

It covers NODE B.



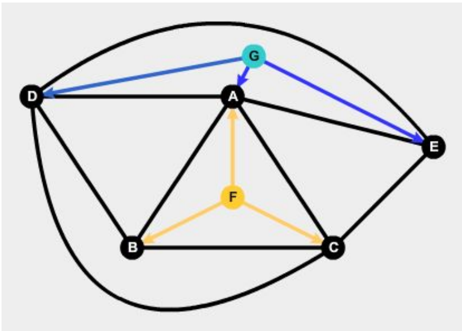


Picture #3

If we were to add one more node, NODE E, on the outer face of the graph (the space around the graph is also counted as a face) it can only connect to NODES D, A, and C in order to stay planar.

What I have come to realize is that all maximum connection planar graphs have only three outermost nodes. An outermost node is a node that can be connected to by adding a node to the outside. For example, in PICTURE #2, when I added NODE D, the outermost nodes became D, C, and A.

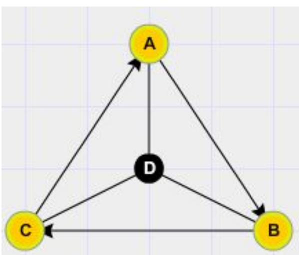
Similar to page 5, you can only connect a node to three other nodes if you are adding the node inside of a face.

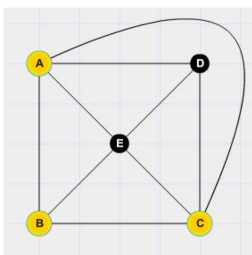


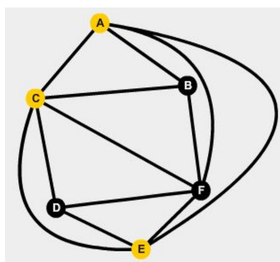
Here is a similar graph to the one on page 5. I have added nodes F and G. They are both placed inside of a face. If we were to place a node inside of triangle ABC, (in this case NODE F), it would only be able to connect to three other nodes because every face, triangle or not, is enclosed by three nodes. Notice that placing a node within the form DAE would only allow the node to connect to the nodes enclosing it.

Examples of Outermost Nodes

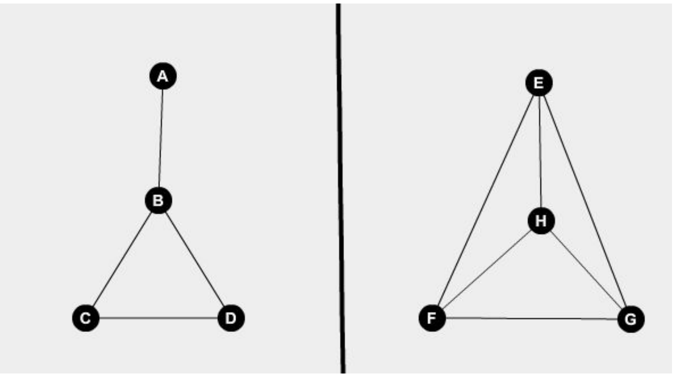
The outermost nodes are highlighted in yellow.







Rule of a planar graph face:
A face created with straight connections that has optimized the possible amount of connections will always be a triangle. This is because every face is enclosed by three nodes. If you have a square face, you can fit one connection across between two nodes.



Not max connections Max connections

Planar vs. Non-Planar
Non-planar graphs that follow the rule $C = 3(n-2)$ and have only 3 connections per node will always be able to be arranged into a planar graph. Note that non-planar graphs that follow this rule will not always have the maximum amount of connections. A max connection non-planar graph cannot be turned into a planar graph because at least one node on that graph will have more than three connections, but this is only true for graphs in which (Nodes = n) $n > 4$.

Relationship of Faces

The relationship between the faces and connections and nodes is something I have yet to find reasoning for, but here is the relationship between faces, connections, and nodes.

$$f = c - (n - 2) \quad \text{or} \quad f - c + n = 2$$

If you are not familiar with counting faces on a planar graph, keep in mind that I counted the space around the graph as a face.

Conclusion

After working through my problem, I have come to a conclusion. The relationship between the number of connections and number of nodes on a planar graph is $C = 3(n-2)$. The reason behind this is the fact that every time you add a node to a planar graph, it will add three additional connections. The arrangement of the nodes does not matter because every arrangement can be moved into any other arrangement provided that the nodes are connected in the same way.