Generalized Polynomial Division with Partial Fractions Residue and Applications to Autonomous Differential Equations

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Abstract

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1 Introduction

Insert introduction

2 Preliminary Lemmas

2.1 Lemma 1

The partial fractions decomposition of $a\left(x\right)=\frac{1}{(x+R_1)^{r_1}(x+R_2)^{r_2}\ldots(x+R_m)^{r_m}}$, where each R_ρ is complex, is

$a\left(x\right)=\frac{1}{\det \mathscr{J}}\sum_{\rho=1}^{m}\left[\sum_{k=1}^{r_\rho}\frac{\text{cof } \mathscr{J}(0,0,\rho,k)}{(x+R_\rho)^k}\right]$, where

$$\mathscr{J}=\begin{bmatrix}
\mathscr{J}(0,0,1,1) & \cdots & \mathscr{J}(0,0,1,r_1) & \mathscr{J}(0,0,2,1) & \cdots & \mathscr{J}(0,0,2,r_2) & \cdots\cdots & \mathscr{J}(0,0,m,1) & \cdots & \mathscr{J}(0,0,m,r_m) \\
\mathscr{J}(1,1,1,1) & \cdots & \mathscr{J}(1,1,1,r_1) & \mathscr{J}(1,1,2,1) & \cdots & \mathscr{J}(1,1,2,r_2) & \cdots\cdots & \mathscr{J}(1,1,m,1) & \cdots & \mathscr{J}(1,1,m,r_m) \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\vdots & & \vdots & \vdots & & \vdots & & \mathscr{J}\left(\ell_{max}(m), \ell_{max}(m), m, 1\right) & \cdots & \mathscr{J}\left(\ell_{max}(m), \ell_{max}(m), m, r_m\right) \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\vdots & & \vdots & \mathscr{J}\left(\ell_{max}(2), \ell_{max}(2), 2, 1\right) & \cdots & \mathscr{J}\left(\ell_{max}(2), \ell_{max}(2), 2, r_2\right) & & \vdots & & \vdots \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\mathscr{J}\left(\ell_{max}(1), \ell_{max}(1), 1, 1\right) & \cdots & \mathscr{J}\left(\ell_{max}(1), \ell_{max}(1), 1, r_1\right) & \vdots & & \vdots & & \vdots & & \vdots \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\mathscr{J}\left(\sum r_\rho - 1, \ell_{max}(1), 1, 1\right) & \cdots & \mathscr{J}\left(\sum r_\rho - 1, \ell_{max}(1), 1, r_1\right) & \mathscr{J}\left(\sum r_\rho - 1, \ell_{max}(2), 2, 1\right) & & \mathscr{J}\left(\sum r_\rho - 1, \ell_{max}(2), 2, r_2\right) & & \mathscr{J}\left(\sum r_\rho - 1, \ell_{max}(m), m, 1\right) & \cdots & \mathscr{J}\left(\sum r_\rho - 1, \ell_{max}(m), m, r_m\right)
\end{bmatrix} \quad (1)$$

and

$$\mathscr{J}\left(y,x,\rho,k\right)=\sum_{\ell=\max\left\{0,y-\sum_{i\neq\rho}r_\rho\right\}}^{x}\left[\binom{r_\rho-k}{\ell}R_\rho^{r_\rho-k-\ell}\sum_{\sum q_i=y-\ell,q_i\in\mathbb{Z}^+}\prod_{i=1,i\neq\rho}^{m}\binom{r_i}{q_i}R_i^{r_i-q_i}\right]. \quad (2)$$

Then we may write

Proof: Partial fractions decomposition (with complex partial fractions coefficients Δ_ℓ^ρ) yields

$$a\left(x\right)=\frac{\Delta_1^1}{x+R_1}+\ldots+\frac{\Delta_{r_1}^1}{(x+R_1)^{r_1}}+\frac{\Delta_1^2}{x+R_2}+\ldots+\frac{\Delta_{r_2}^2}{(x+R_2)^{r_2}}+\ldots\ldots\ldots+\frac{\Delta_1^m}{x+R_m}+\ldots+\frac{\Delta_{r_m}^m}{(x+R_m)^{r_m}}. \quad (3)$$

We see that the partial fractions coefficients in (1) must satisfy the constraint equation $\frac{1}{\prod_{i=1}^{m}(x+R_i)^{r_i}} = \sum_{\rho=1}^{m}\left[\sum_{\ell=1}^{r_\rho}\frac{\Delta_\ell^\rho}{(x+R_\rho)^\ell}\right]$. Multiplying by the LHS product to cancel the denominator in the innermost RHS sum, we see that the constraint equation is equivalent to $1 = \sum_{\rho=1}^{m}\left\{\sum_{\ell=1}^{r_\rho}\left[\Delta_\ell^\rho\left(x+R_\rho\right)^{r_\rho-\ell}\prod_{i=1,i\neq\rho}^{m}\left(x+R_i\right)^{r_i}\right]\right\}$ (with $x \neq R_i$), which, upon factoring, becomes

$$1 = \sum_{\rho=1}^{m}\left\{\left[\sum_{\ell=1}^{r_\rho}\Delta_\ell^\rho\left(x+R_\rho\right)^{r_\rho-\ell}\right]\left[\prod_{i=1,i\neq\rho}^{m}\left(x+R_i\right)^{r_i}\right]\right\}. \tag{4}$$

Binomial expansion of the sum yields $\sum_{\ell=1}^{r_\rho}\Delta_\ell^\rho\left(x+R_\rho\right)^{r_\rho-\ell} = \sum_{\ell=1}^{r_\rho}\Delta_\ell^\rho\left(\sum_{j=0}^{r_\rho-\ell}\binom{r_\rho-\ell}{j}R_\rho^{r_\rho-\ell-j}x^j\right)$, which has an x^h coefficient $\sigma\left(h,\rho\right) = \sum_{k=1}^{r_\rho-h}\Delta_k^\rho\binom{r_\rho-k}{h}R_\rho^{r_\rho-k-h}$, where $0 \leq h \leq r_\rho - 1$.

Binomial expansion of the product yields $\prod_{i=1,i\neq\rho}^{m}\left(x+R_i\right)^{r_i} = \prod_{i=1,i\neq\rho}^{m}\left(\sum_{e=0}^{r_i}\binom{r_i}{e}R_i^{r_i-e}x^e\right)$, which has an x^δ coefficient $\lambda\left(\delta,\rho,m\right) = \sum_{\sum q_i=\delta,q_i\in\mathbb{Z}^+}\left[\prod_{i=1,i\neq\rho}^{m}\binom{r_i}{q_i}R_i^{r_i-q_i}\right]$, where $0 \leq \delta \leq \sum_{i\neq\rho}r_i$.

Substitution of $\sigma\left(h,\rho\right)$ and $\lambda\left(\delta,\rho,m\right)$ in (2) yields

$$1 = \sum_{\rho=1}^{m}\left\{\left[\sum_{h=1}^{r_\rho-1}\sigma\left(h,\rho\right)x^h\right]\left[\sum_{\delta=0}^{\sum_{i\neq\rho}r_i}\lambda\left(\delta,\rho,m\right)x^\delta\right]\right\}. \tag{5}$$

We see the x^α coefficient of $\left[\sum_{h=1}^{r_\rho-1}\sigma\left(h,\rho,m\right)x^h\right]\left[\sum_{\delta=0}^{\sum_{i\neq\rho}r_i}\lambda\left(\delta,\rho,m\right)x^\delta\right]$ is $\sum_{\beta=0}^{\alpha}\sigma\left(\beta,\rho,m\right)g\left(\alpha-\beta,\rho,m\right)$, where $0 \leq \alpha \leq \sum_{\rho=1}^{k}r_\rho - 1$.

Thus, (3) becomes

$$1 = \sum_{\rho=1}^{m}\left\{\sum_{\alpha=0}^{\sum r_\rho-1}\left[\sum_{\beta=0}^{\alpha}\sigma\left(\beta,\rho\right)\lambda\left(\alpha-\beta,\rho,m\right)\right]x^\alpha\right\}. \tag{6}$$

Note that $\sigma\left(h,\rho\right) = 0$ when $h > r_\rho - 1$ and $\lambda\left(\delta,\rho,m\right) = 0$ when $\delta > \sum i \neq \rho r_i$, as they will be useful later.

Knowing that the RHS of (4) must equal 1 when $\alpha = 0$ and must vanish when $\alpha \neq 0$, we write the system

$$\begin{aligned}
1 &= \sum_{\rho=1}^{m}\left[\lambda\left(0,\rho,m\right)\sigma\left(0,\rho\right)\right] \\
0 &= \sum_{\rho=1}^{m}\left[\lambda\left(1,\rho,m\right)\sigma\left(0,\rho\right)+\lambda\left(0,\rho,m\right)\sigma\left(1,\rho\right)\right] \\
&\vdots \\
0 &= \sum_{\rho=1}^{m}\left[\lambda\left(\sum r_\rho-1,\rho,m\right)\sigma\left(0,\rho\right)+\lambda\left(\sum r_\rho-2,\rho,m\right)\sigma\left(1,\rho\right)+...+\lambda\left(0,\rho,m\right)\sigma\left(\sum r_\rho-1,\rho,m\right)\right],
\end{aligned} \tag{7}$$

or as a matrix,

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda(0,1,m) & \cdots & \lambda(0,m,m) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda(1,1,m) & \cdots & \lambda(1,m,m) & \lambda(0,1,m) & \cdots & \lambda(0,m,m) & 0 & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \lambda\left(\sum r_\rho - 1, 1, m\right) & \cdots & \lambda\left(\sum r_\rho - 1, m, m\right) & \lambda\left(\sum r_\rho - 2, 1, m\right) & \cdots & \lambda\left(\sum r_\rho - 2, m, m\right) & \cdots\cdots\cdots & \lambda(0,1,m) & \cdots & \lambda(0,m,m) \end{bmatrix} \begin{bmatrix} \sigma(0,1) \\ \vdots \\ \sigma(0,m) \\ \sigma(1,1) \\ \vdots \\ \sigma(1,m) \\ \vdots \\ \vdots \\ \sigma\left(\sum r_\rho - 1, 1\right) \\ \vdots \\ \sigma\left(\sum r_\rho - 1, m\right) \end{bmatrix} \tag{8}$$

or $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda\sigma.$

We will now decompose σ: Let $\varphi(\beta, \rho, k) = \binom{r_\rho - k}{\beta} R_\rho^{r_\rho - k - \beta}$, so that $\sigma(\beta, \rho) = \sum_{k=1}^{r_\rho - \beta} \varphi(\beta, \rho, k)\, \Delta_k^\rho$ (as evidenced by previous proof, note that $\varphi(\beta, \rho, k) = 0$ when $\beta > r_\rho - 1$). Then we have the system

$$\begin{aligned} \sigma(0,1) &= \sum_{k=1}^{r_1 - 0} \varphi(0,1,k)\, \Delta_k^1 \\ &\vdots \\ \sigma(0,m) &= \sum_{k=1}^{r_m - 0} \varphi(0,m,k)\, \Delta_k^m \\ \sigma(1,1) &= \sum_{k=1}^{r_1 - 1} \varphi(1,1,k)\, \Delta_k^1 \\ &\vdots \\ \sigma(1,m) &= \sum_{k=1}^{r_m - 1} \varphi(1,m,k)\, \Delta_k^m \\ &\vdots \\ &\vdots \\ \sigma\left(\sum r_\rho - 1, 1\right) &= \sum_{k=1}^{r_1 - \left(\sum r_\rho - 1\right)} \varphi\left(\left(\sum r_\rho - 1\right), 1, k\right) \Delta_k^1 \\ &\vdots \\ \sigma\left(\left(\sum r_\rho - 1\right), m\right) &= \sum_{k=1}^{r_m - \left(\sum r_\rho - 1\right)} \varphi\left(\left(\sum r_\rho - 1\right), m, k\right) \Delta_k^m \end{aligned} \tag{9}$$

or, as a matrix,

$$
\sigma = \begin{bmatrix}
\varphi(0,1,1) & \cdots & \varphi(0,1,r_1) & - & - & 0 & - \\
\vdots & & \vdots & & & & \\
- & 0 & - & - & \varphi(0,m,1) & \cdots & \varphi(0,m,r_m) \\
\varphi(1,1,1) & \cdots & \varphi(1,1,r_1) & - & - & 0 & - \\
\vdots & & \vdots & & & & \\
- & 0 & - & - & \varphi(1,m,1) & \cdots & \varphi(0,m,r_m) \\
 & & & & \vdots & & \vdots \\
 & & & & \vdots & & \vdots \\
\varphi\left(\left(\sum r_\rho - 1\right),1,1\right) & \cdots & \varphi\left(\left(\sum r_\rho - 1\right),1,r_1\right) & - & - & 0 & - \\
\vdots & & \vdots & & & & \\
- & 0 & - & - & \varphi\left(\left(\sum r_\rho - 1\right),m,1\right) & \cdots & \varphi\left(\left(\sum r_\rho - 1\right),m,r_m\right)
\end{bmatrix}
\begin{bmatrix}
\Delta_1^1 \\ \vdots \\ \Delta_{r_1}^1 \\ \Delta_2^1 \\ \vdots \\ \Delta_{r_2}^2 \\ \vdots \\ \vdots \\ \Delta_1^m \\ \vdots \\ \Delta_{r_m}^m
\end{bmatrix}
\tag{10}
$$

or $\sigma = \varphi\Delta$. Then we have $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda\varphi\Delta$. Let $\mathscr{J} = \lambda\varphi$, so that $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathscr{J}\Delta$. We see that

$$
\mathscr{J}=\begin{bmatrix}
\begin{bmatrix}\lambda(0,1,m)\\ *\varphi(0,1,1)\end{bmatrix} & \cdots & \begin{bmatrix}\lambda(0,1,m)\\ *\varphi(0,1,r_1)\end{bmatrix} & \begin{bmatrix}\lambda(0,2,m)\\ *\varphi(0,2,1)\end{bmatrix} & \cdots & \begin{bmatrix}\lambda(0,2,m)\\ *\varphi(0,2,r_2)\end{bmatrix} & \cdots\cdots & \begin{bmatrix}\lambda(0,m,m)\\ *\varphi(0,m,1)\end{bmatrix} & \cdots & \begin{bmatrix}\lambda(0,m,m)\\ *\varphi(0,m,r_m)\end{bmatrix}\\
\begin{bmatrix}\lambda(1,1,m)\\ *\varphi(0,1,1)\\ +\lambda(0,1,m)\\ *\varphi(1,1,1)\end{bmatrix} & \cdots & \begin{bmatrix}\lambda(1,1,m)\\ *\varphi(0,1,r_1)\\ +\lambda(0,1,m)\\ *\varphi(1,1,r_1)\end{bmatrix} & \begin{bmatrix}\lambda(1,2,m)\\ *\varphi(0,2,1)\\ +\lambda(0,2,m)\\ *\varphi(1,2,1)\end{bmatrix} & \cdots & \begin{bmatrix}\lambda(1,2,m)\\ *\varphi(0,2,r_2)\\ +\lambda(0,2,m)\\ *\varphi(1,2,r_2)\end{bmatrix} & \cdots\cdots & \begin{bmatrix}\lambda(1,m,m)\\ *\varphi(0,m,1)\\ +\lambda(0,m,m)\\ *\varphi(1,m,1)\end{bmatrix} & \cdots & \begin{bmatrix}\lambda(1,m,m)\\ *\varphi(0,m,r_m)\\ +\lambda(0,m,m)\\ *\varphi(1,m,r_m)\end{bmatrix}\\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots\\
\begin{bmatrix}\lambda\left(\left(\sum r_\rho-1\right),1,m\right)\\ *\varphi(0,1,1)\\ +\lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -1\end{smallmatrix},1,m\right)\\ *\varphi(1,1,1)\\ +\cdots\\ +\lambda(0,1,m)\\ *\varphi\left(\left(\sum r_\rho-1\right),1,1\right)\end{bmatrix} & \cdots & \begin{bmatrix}\lambda\left(\left(\sum r_\rho-1\right),1,m\right)\\ *\varphi(0,1,r_1)\\ +\lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -1\end{smallmatrix},1,m\right)\\ *\varphi(1,1,r_1)\\ +\cdots\\ +\lambda(0,1,m)\\ *\varphi\left(\left(\sum r_\rho-1\right),1,r_1\right)\end{bmatrix} & \begin{bmatrix}\lambda\left(\left(\sum r_\rho-1\right),2,m\right)\\ *\varphi(0,2,1)\\ +\lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -1\end{smallmatrix},2,m\right)\\ *\varphi(1,2,1)\\ +\cdots\\ +\lambda(0,2,m)\\ *\varphi\left(\left(\sum r_\rho-1\right),2,1\right)\end{bmatrix} & \cdots & \begin{bmatrix}\lambda\left(\left(\sum r_\rho-1\right),2,m\right)\\ *\varphi(0,2,r_2)\\ +\lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -1\end{smallmatrix},2,m\right)\\ *\varphi(1,2,r_2)\\ +\cdots\\ +\lambda(0,2,m)\\ *\varphi\left(\left(\sum r_\rho-1\right),2,r_2\right)\end{bmatrix} & \cdots\cdots & \begin{bmatrix}\lambda\left(\left(\sum r_\rho-1\right),m,m\right)\\ *\varphi(0,m,1)\\ +\lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -1\end{smallmatrix},m,m\right)\\ *\varphi(1,m,1)\\ +\cdots\\ \lambda(0,m,m)\\ *\varphi\left(\left(\sum r_\rho-1\right),m,1\right)\end{bmatrix} & \cdots & \begin{bmatrix}\lambda\left(\left(\sum r_\rho-1\right),m,m\right)\\ *\varphi(0,m,r_m)\\ +\lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -1\end{smallmatrix},m,m\right)\\ *\varphi(1,m,r_m)\\ +\cdots\\ +\lambda(0,m,m)\\ *\varphi\left(\left(\sum r_\rho-1\right),m,r_m\right)\end{bmatrix}
\end{bmatrix} \tag{11}
$$

or, more compactly,

$$
\begin{bmatrix}
\begin{bmatrix}\sum_{\ell=0}^{0}\\ \lambda(0-\ell,1,m)\\ *\varphi(\ell,1,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=0}^{0}\\ \lambda(0-\ell,1,m)\\ *\varphi(\ell,1,r_1)\end{bmatrix} & \begin{bmatrix}\sum_{\ell=0}^{0}\\ \lambda(0-\ell,2,m)\\ *\varphi(\ell,2,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=0}^{0}\\ \lambda(0-\ell,2,m)\\ *\varphi(\ell,2,r_2)\end{bmatrix} & \cdots\cdots & \begin{bmatrix}\sum_{\ell=0}^{0}\lambda(0-\ell,m,m)\\ *\varphi(\ell,m,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=0}^{0}\\ \lambda(0-\ell,m,m)\\ *\varphi(\ell,m,r_m)\end{bmatrix}\\
\begin{bmatrix}\sum_{\ell=0}^{1}\\ \lambda(1-\ell,1,m)\\ *\varphi(\ell,1,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=0}^{1}\\ \lambda(1-\ell,1,m)\\ *\varphi(\ell,1,r_1)\end{bmatrix} & \begin{bmatrix}\sum_{\ell=0}^{1}\\ \lambda(1-\ell,2,m)\\ *\varphi(\ell,2,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=0}^{1}\\ \lambda(1-\ell,2,m)\\ *\varphi(\ell,2,r_2)\end{bmatrix} & \cdots\cdots & \begin{bmatrix}\sum_{\ell=0}^{1}\\ \lambda(1-\ell,m,m)\\ *\varphi(\ell,m,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=0}^{1}\\ \lambda(1-\ell,m,m)\\ *\varphi(\ell,m,r_m)\end{bmatrix}\\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots\\
\begin{bmatrix}\sum_{\ell=0}^{(\sum r_\rho-1)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},1,m\right)\\ *\varphi(\ell,1,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=0}^{(\sum r_\rho-1)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},1,m\right)\\ *\varphi(\ell,1,r_1)\end{bmatrix} & \begin{bmatrix}\sum_{\ell=0}^{(\sum r_\rho-1)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},2,m\right)\\ *\varphi(\ell,2,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=0}^{(\sum r_\rho-1)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},2,m\right)\\ *\varphi(\ell,2,r_2)\end{bmatrix} & \cdots\cdots & \begin{bmatrix}\sum_{\ell=0}^{(\sum r_\rho-1)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},m,m\right)\\ *\varphi(\ell,m,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=0}^{(\sum r_\rho-1)}\lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},m,m\right)\\ *\varphi(\ell,m,r_m)\end{bmatrix}
\end{bmatrix} \tag{12}
$$

Recalling that $\varphi(\beta,\rho,k)=0$ when $\beta>r_\rho-1$ and $\lambda(\beta,\rho,m)=0$ when $\beta>\sum_{i\neq\rho}r_i$, and letting $\ell_{min}(\rho,y)=max\left\{0,y-\sum_{i\neq\rho}r_i\right\}$ and $\ell_{max}=(\rho)=r_\rho-1$, we see that $\mathscr{J}$ simplifies to

$$
\left[\begin{array}{ccccccccccc}
\begin{bmatrix}\sum_{\ell=\ell_{min}(1,0)}^{\ell_{max}(1)}\\ \lambda(0-\ell,1,m)\\ *\varphi(\ell,1,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(1,0)}^{\ell_{max}(1)}\\ \lambda(0-\ell,1,m)\\ *\varphi(\ell,1,r_1)\end{bmatrix} & \begin{bmatrix}\sum_{\ell=\ell_{min}(2,0)}^{\ell_{max}(2)}\\ \lambda(0-\ell,2,m)\\ *\varphi(\ell,2,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(2,0)}^{\ell_{max}(2)}\\ \lambda(0-\ell,2,m)\\ *\varphi(\ell,2,r_2)\end{bmatrix} & \cdots\cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(m,0)}^{\ell_{max}(m)}\\ \lambda(0-\ell,m,m)\\ *\varphi(\ell,m,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(m,0)}^{\ell_{max}(m)}\\ \lambda(0-\ell,m,m)\\ *\varphi(\ell,m,r_m)\end{bmatrix} \\
\begin{bmatrix}\sum_{\ell=\ell_{min}(1,1)}^{\ell_{max}(1)}\\ \lambda(1-\ell,1,m)\\ *\varphi(\ell,1,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(1,1)}^{\ell_{max}(1)}\\ \lambda(1-\ell,1,m)\\ *\varphi(\ell,1,r_1)\end{bmatrix} & \begin{bmatrix}\sum_{\ell=\ell_{min}(2,1)}^{\ell_{max}(2)}\\ \lambda(1-\ell,2,m)\\ *\varphi(\ell,2,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(2,1)}^{\ell_{max}(2)}\\ \lambda(1-\ell,2,m)\\ *\varphi(\ell,2,r_2)\end{bmatrix} & \cdots\cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(m,1)}^{\ell_{max}(m)}\\ \lambda(1-\ell,m,m)\\ *\varphi(\ell,m,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(m,1)}^{\ell_{max}(m)}\\ \lambda(1-\ell,m,m)\\ *\varphi(\ell,m,r_m)\end{bmatrix} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & & \begin{bmatrix}\sum_{\ell=\ell_{min}(m,\sum r_\rho-1)}^{\ell_{max}(m)}\\ \lambda\left(\begin{smallmatrix}\ell_{max}(m)\\ -\ell\end{smallmatrix},m,m\right)\\ *\varphi(\ell,m,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(m,\sum r_\rho-1)}^{\ell_{max}(m)}\\ \lambda\left(\begin{smallmatrix}\ell_{max}(m)\\ -\ell\end{smallmatrix},m,m\right)\\ *\varphi(\ell,m,r_m)\end{bmatrix} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & \begin{bmatrix}\sum_{\ell=\ell_{min}(2,\sum r_\rho-1)}^{\ell_{max}(2)}\\ \lambda\left(\begin{smallmatrix}\ell_{max}(2)\\ -\ell\end{smallmatrix},2,m\right)\\ *\varphi(\ell,2,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(2,\sum r_\rho-1)}^{\ell_{max}(2)}\\ \lambda\left(\begin{smallmatrix}\ell_{max}(2)\\ -\ell\end{smallmatrix},2,m\right)\\ *\varphi(\ell,2,r_2)\end{bmatrix} & & & & \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\begin{bmatrix}\sum_{\ell=\ell_{min}(1,\sum r_\rho-1)}^{\ell_{max}(1)}\\ \lambda\left(\begin{smallmatrix}\ell_{max}(1)\\ -\ell\end{smallmatrix},1,m\right)\\ *\varphi(\ell,1,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(1,\sum r_\rho-1)}^{\ell_{max}(1)}\\ \lambda\left(\begin{smallmatrix}\ell_{max}(1)\\ -\ell\end{smallmatrix},1,m\right)\\ *\varphi(\ell,1,r_1)\end{bmatrix} & & & & & & & \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\begin{bmatrix}\sum_{\ell=\ell_{min}(1,\sum r_\rho-1)}^{\ell_{max}(1)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},1,m\right)\\ *\varphi(\ell,1,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(1,\sum r_\rho-1)}^{\ell_{max}(1)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},1,m\right)\\ *\varphi(\ell,1,r_1)\end{bmatrix} & \begin{bmatrix}\sum_{\ell=\ell_{min}(2,\sum r_\rho-1)}^{\ell_{max}(2)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},2,m\right)\\ *\varphi(\ell,2,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(2,\sum r_\rho-1)}^{\ell_{max}(2)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},2,m\right)\\ *\varphi(\ell,2,r_2)\end{bmatrix} & \cdots\cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(m,\sum r_\rho-1)}^{\ell_{max}(m)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},m,m\right)\\ *\varphi(\ell,m,1)\end{bmatrix} & \cdots & \begin{bmatrix}\sum_{\ell=\ell_{min}(m,\sum r_\rho-1)}^{\ell_{max}(m)}\\ \lambda\left(\begin{smallmatrix}(\sum r_\rho-1)\\ -\ell\end{smallmatrix},m,m\right)\\ *\varphi(\ell,m,r_m)\end{bmatrix}
\end{array}\right] \quad (13)
$$

Let

$$\mathscr{J}\left(y,x,\rho,k\right)=\sum_{\ell=\ell_{min}(\rho,y)}^{x}\lambda\left(y-\ell,\rho,m\right)\varphi\left(\ell,\rho,k\right) \tag{14}$$

$$=\sum_{\ell=\max\left\{0,y-\sum_{i\neq\rho}r_{\rho}\right\}}^{x}\left\{\left[\sum_{\sum q_i=y-\ell,q_i\in\mathbb{Z}^+}\left(\prod_{i=1,i\neq\rho}^{m}\binom{r_i}{q_i}R_i^{r_i-q_i}\right)\right]\left[\binom{r_\rho-k}{\ell}R_\rho^{r_\rho-k-\ell}\right]\right\}. \tag{15}$$

Then we may write

$$\mathscr{J}=\begin{bmatrix}
\mathscr{J}(0,0,1,1) & \cdots & \mathscr{J}(0,0,1,r_1) & \mathscr{J}(0,0,2,1) & \cdots & \mathscr{J}(0,0,2,r_2) & \cdots\cdots & \mathscr{J}(0,0,m,1) & \cdots & \mathscr{J}(0,0,m,r_m)\\
\mathscr{J}(1,1,1,1) & \cdots & \mathscr{J}(1,1,1,r_1) & \mathscr{J}(1,1,2,1) & \cdots & \mathscr{J}(1,1,2,r_2) & \cdots\cdots & \mathscr{J}(1,1,m,1) & \cdots & \mathscr{J}(1,1,m,r_m)\\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots\\
\vdots & & \vdots & \vdots & & \vdots & & \mathscr{J}\binom{\ell_{max}(m),}{\ell_{max}(m),m,1} & \cdots & \mathscr{J}\binom{\ell_{max}(m),}{\ell_{max}(m),m,r_m}\\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots\\
\vdots & & \vdots & \mathscr{J}\binom{\ell_{max}(2),}{\ell_{max}(2),2,1} & \cdots & \mathscr{J}\binom{\ell_{max}(2),}{\ell_{max}(2),2,r_2} & & \vdots & & \vdots\\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots\\
\mathscr{J}\binom{\ell_{max}(1),}{\ell_{max}(1),1,1} & \cdots & \mathscr{J}\binom{\ell_{max}(1),}{\ell_{max}(1),1,r_1} & \vdots & & \vdots & & \vdots & & \vdots\\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots\\
\mathscr{J}\binom{\sum r_\rho-1,}{\ell_{max}(1),1,1} & \cdots & \mathscr{J}\binom{\sum r_\rho-1,}{\ell_{max}(1),1,r_1} & \mathscr{J}\binom{\sum r_\rho-1,}{\ell_{max}(2),2,1} & & \mathscr{J}\binom{\sum r_\rho-1,}{\ell_{max}(2),2,r_2} & & \mathscr{J}\binom{\sum r_\rho-1,}{\ell_{max}(m),m,1} & \cdots & \mathscr{J}\binom{\sum r_\rho-1,}{\ell_{max}(m),m,r_m}
\end{bmatrix} \tag{16}$$

Because partial fractions decomposition has been shown to be unique, we know $\mathscr{J}$ must be invertible. Thus, $\Delta=\mathscr{J}^{-1}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$, which is the first column of $\mathscr{J}^{-1}$. But we also know that $\mathscr{J}^{-1}=\frac{1}{\det\mathscr{J}}\text{cof}\left(\mathscr{J}\right)^T$, so $\Delta=\frac{1}{\det\mathscr{J}}\cdot$ first column of cof $\left(\mathscr{J}\right)^T=\frac{1}{\det\mathscr{J}}\cdot\left[\text{first row of cof }\left(\mathscr{J}\right)\right]^T$. So then we have

$$\begin{bmatrix}\Delta_1^1\\\vdots\\\Delta_{r_1}^1\\\Delta_1^2\\\vdots\\\Delta_{r_2}^2\\\vdots\\\vdots\\\Delta_1^m\\\vdots\\\Delta_{r_m}^m\end{bmatrix}=\frac{1}{\det\mathscr{J}}\begin{bmatrix}\text{cof }\mathscr{J}\left(0,0,1,1\right)\\\vdots\\\text{cof }\mathscr{J}\left(0,0,1,r_1\right)\\\text{cof }\mathscr{J}\left(0,0,2,1\right)\\\vdots\\\text{cof }\mathscr{J}\left(0,0,2,r_2\right)\\\vdots\\\vdots\\\text{cof }\mathscr{J}\left(0,0,m,1\right)\\\vdots\\\text{cof }\mathscr{J}\left(0,0,m,r_m\right)\end{bmatrix} \tag{17}$$

So then

$$\Delta_k^\rho=\frac{1}{\det\mathscr{J}}\text{cof }\mathscr{J}\left(0,0,\rho,k\right). \tag{18}$$

Thus,

$$a\left(x\right)=\frac{1}{\det\mathscr{J}}\left(\frac{\operatorname{cof}\mathscr{J}\left(0,0,1,1\right)}{x+R_1}+\ldots+\frac{\operatorname{cof}\mathscr{J}\left(0,0,1,r_1\right)}{\left(x+R_1\right)^{r1}}+\frac{\operatorname{cof}\mathscr{J}\left(0,0,2,1\right)}{x+R_2}+\ldots+\frac{\operatorname{cof}\mathscr{J}\left(0,0,2,r_2\right)}{\left(x+R_2\right)^{r2}}+\ldots\ldots\ldots+\frac{\operatorname{cof}\mathscr{J}\left(0,0,m,1\right)}{x+R_m}+\ldots+\frac{\operatorname{cof}\mathscr{J}\left(0,0,m,,r_m\right)}{\left(x+R_m\right)^{rm}}\right). \tag{19}$$

or more compactly, $a\left(x\right)=\frac{1}{\det\mathscr{J}}\sum_{\rho=1}^{m}\left[\sum_{k=1}^{r_\rho}\frac{\operatorname{cof}\mathscr{J}\left(0,0,\rho,k\right)}{\left(x+R_\rho\right)^k}\right]$.

2.2 Lemma 2

$$\frac{x^m}{\left(x+a\right)^{\alpha}}=\sum_{i=0}^{m-\alpha}\binom{m-1-i}{\alpha-1}\left(-a\right)^{m-\alpha-i}x^i+\sum_{i=\max\{\alpha-m,1\}}^{\alpha}\frac{\binom{m}{\alpha-i}\left(-a\right)^{m-\alpha+i}}{\left(x+a\right)^i} \tag{20}$$

$\forall\ m,\alpha\in\mathbb{N}$, with $x\neq -a$, and where the binomial coefficients are taken to be 0 where they are otherwise undefined.

Proof: We proceed by double induction, abbreviating the above as $P\left(m,\alpha\right)$.

First Base Case: We prove $P(1,1)$ as the basis for the first induction:

$$\frac{x}{x+a}=\sum_{i=0}^{0}\binom{-i}{0}\left(-a\right)^{-i}x^i+\sum_{i=\max\ \{0,1\}}\frac{\binom{1}{1-i}\left(-a\right)^i}{\left(x+a\right)^i}=1+\frac{-a}{x+a}=\frac{x}{x+a}. \tag{21}$$

First Inductive Step: Assume $P(m,1)$ for $m\in\mathbb{N}$. We will show that $P(m+1,1)$ follows. Using $P(m,1)$, we write $\frac{x^{m+1}}{x+a}=xp(m,1)+xf(m,1)$. But we can also see that $p(m,1)=\sum_{i=0}^{m-1}\binom{m-1-i}{0}\left(-a\right)^{m-1-i}x^i$ and $f(m,1)=\frac{(-a)^m}{x+a}$. Substituting, we see

$$\frac{x^{m+1}}{x+a}=\sum_{i=0}^{m-1}\binom{m-1-i}{0}\left(-a\right)^{m-1-i}x^{i+1}+\frac{\left(-a\right)^m x}{x+a}=\sum_{i=1}^{m}\binom{m-1}{0}\left(-a\right)^{m-i}x^i+\left(-a\right)^m+\frac{\left(-a\right)^{m+1}}{x+a} \tag{22}$$

$$=\sum_{i=0}^{m}\binom{m-i}{0}\left(-a\right)^{m-i}x^i+\frac{\left(-a\right)^{m+1}}{x+a}=p(m+1,1)+f(m+1,1)$$

First Inductive Conclusion (Second Base Case): We have proven $P(1,1)$ and shown $P(m,1)\implies P(m+1,1)$ for $m\in\mathbb{N}$. Therefore, $P(m,1)\ \forall\ m\in\mathbb{N}$.

Second Inductive Step: Assume $P(m,\alpha)$ for $m,\alpha\in\mathbb{N}$. We will show that $P(m,\alpha+1)$ follows. Using $P(m,\alpha)$, we write $\frac{x^m}{(x+a)^{\alpha+1}}=\frac{p(m,\alpha)}{x+a}+\frac{f(m,\alpha)}{x+a}$. We see that

$$\frac{f(m,\alpha)}{x+a}=\sum_{i=\max\ \{\alpha-m,1\}}^{\alpha}\frac{\binom{m}{\alpha-i}\left(-a\right)^{m-\alpha+i}}{\left(x+a\right)^{i+1}}=\sum_{i=1}^{\alpha}\frac{\binom{m}{\alpha-i}\left(-a\right)^{m-\alpha+i}}{\left(x+a\right)^i}\ \text{(introducing 0 terms)} \tag{23}$$

$$=\sum_{i=2}^{\alpha+1}\frac{\binom{m}{\alpha+1-i}\left(-a\right)^{m-\alpha-1+i}}{\left(x+a\right)^i}=\sum_{i=1}^{\alpha+1}\frac{\binom{m}{\alpha+1-i}\left(-a\right)^{m-\alpha-1+i}}{\left(x+a\right)^i}-\frac{\binom{m}{\alpha}\left(-a\right)^{m-\alpha}}{x+a} \tag{24}$$

$$=f(m,\alpha+1)-\frac{\binom{m}{\alpha}\left(-a\right)^{m-\alpha}}{x+a}\ \text{(removing 0 terms)} \tag{25}$$

Using Pascal's identity we see $\binom{m-1-i}{\alpha-1} = \binom{m-i}{\alpha} - \binom{m-1-i}{\alpha}$, so

$$p(m,\alpha) = \sum_{i=0}^{m-\alpha} \binom{m-i}{\alpha} (-a)^{m-\alpha-i} x^i - \sum_{i=0}^{m-\alpha} \binom{m-1-i}{\alpha} (-a)^{m-\alpha-i} x^i \tag{26}$$

$$= \sum_{i=-1}^{m-\alpha-1} \binom{m-1-i}{\alpha} (-a)^{m-\alpha-1-i} + a \sum_{i=0}^{m-\alpha} \binom{m-1-i}{\alpha} (-a)^{m-\alpha-1-i} x^i \tag{27}$$

$$= \binom{m}{\alpha} (-a)^{m-\alpha} + x \sum_{i=0}^{m-\alpha-1} \binom{m-1-i}{\alpha} (-a)^{m-\alpha-1-i} x^i + a \binom{\alpha-1}{\alpha} (-a)^{-1} x^{m-\alpha} + a \sum_{i=0}^{m-\alpha-1} \binom{m-1-i}{\alpha} (-a)^{m-\alpha-1-i} x^i \tag{28}$$

$$= \binom{m}{\alpha} (-a)^{m-\alpha} + (x+a)\, p(m,\alpha+1) \tag{29}$$

Substituting in our first equation, we see

$$\frac{x^m}{(x+a)^{\alpha+1}} = \frac{\binom{m}{\alpha} (-a)^{m-\alpha} + (x+a)\, p(m,\alpha+1)}{x+a} + f(m,\alpha+1) - \frac{\binom{m}{\alpha} (-a)^{m-\alpha}}{x+a} = p(m,\alpha+1) + f(m,\alpha+1) \tag{30}$$

Second Inductive Conclusion: We have proven $P(m,1)$ and shown $P(m,\alpha) \implies P(m,\alpha+1)$ for $m,\alpha \in \mathbb{N}$. Therefore, $P(m,\alpha)\ \forall\ m,\alpha \in \mathbb{N}$.

3 Proof

Let $F(x) = \frac{N(x)}{D(x)} = \frac{N_0+N_1x+N_2x^2+\ldots+N_nx^n}{D_0+D_1x+D_2x^2+\ldots+D_dx^d}$ where every numerator coefficient N and denominator coefficient D is real. By the fundamental theorem of algebra, we see that $D(x) = (x+R_1)^{r_1}(x+R_2)^{r_2}\ldots(x+R_m)^{r_m} = \prod_{i=1}^{m}(x+R_i)^{r_i}$ for some complex roots $-R$. By Lemma 1, $F(x) = \frac{N(x)}{\det \mathscr{J}} \sum_{\rho=1}^{m}\left[\sum_{k=1}^{r_\rho} \frac{\operatorname{cof} \mathscr{J}_{(0,0,\rho,k)}}{(x+R_\rho)^k}\right]$.

By Lemma 2, we have

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$$\frac{x^m}{(x+R_1)^{r_1}(x+R_2)^{r_2}\cdots(x+R_k)^{r_k}} = \sum_{\rho=1}^{k}\left[\sum_{\ell=1}^{r_\rho} \Delta_\ell^\rho \frac{x^m}{(x+R_\rho)^\ell}\right] \tag{31}$$

$$= \sum_{\rho=1}^{k}\left[\sum_{\ell=1}^{r_\rho} \Delta_\ell^\rho \left[\sum_{i=0}^{m-\ell} \binom{m-1-i}{\ell-1} (-R_\rho)^{m-\ell-i} x^i + \sum_{i=1}^{\ell} \frac{\binom{m}{\ell-i} (-R_\rho)^{m-\ell+i}}{(x+R_\rho)^i}\right]\right].$$

Expanding and regrouping, we see that

$$\sum_{\rho=1}^{k}\left[\sum_{\ell=1}^{r_\rho} \Delta_\ell^\rho \left[\sum_{i=0}^{m-\ell} \binom{m-1-i}{\ell-1} (-R_\rho)^{m-\ell-i} x^i\right]\right] = \sum_{\rho=1}^{k}\left\{\sum_{\zeta=0}^{m-1}\left[\sum_{\ell=1}^{m-\zeta} \Delta_\ell^\rho \binom{m-1-\zeta}{\ell-1} (-R_\rho)^{m-\zeta-\ell}\right] x^\zeta\right\} \tag{32}$$

$$= \sum_{\zeta=0}^{m-1}\left\{\sum_{\rho=1}^{k}\left[\sum_{\ell=1}^{m-\zeta} \Delta_\ell^\rho \binom{m-1-\zeta}{\ell-1} (-R_\rho)^{m-\zeta-\ell}\right]\right\} x^\zeta$$

and also

$$\sum_{\rho=1}^{k}\left[\sum_{\ell=1}^{r_\rho} \Delta_\ell^\rho \left[\sum_{i=1}^{\ell} \frac{\binom{m}{\ell-i} (-R_\rho)^{m-\ell+i}}{(x+R_\rho)^i}\right]\right] = \sum_{\rho=1}^{k}\left\{\sum_{\zeta=1}^{r_\rho} \frac{\left[\sum_{\ell=s}^{r_\rho} \Delta_\ell^\rho \binom{m}{\ell-1} (-R_\rho)^{m+1-\ell}\right]}{(x+R_\rho)^\zeta}\right\} \tag{33}$$

so then

$$\frac{x^m}{(x+R_1)^{r_1}(x+R_2)^{r_2}\cdots(x+R_k)^{r_k}} = \sum_{\zeta=0}^{m-1}\left\{\sum_{\rho=1}^{k}\left[\sum_{\ell=1}^{m-\zeta}\Delta_\ell^\rho\binom{m-1-\zeta}{\ell-1}(-R_\rho)^{m-\zeta-\ell}\right]\right\}x^\zeta \\ +\sum_{\rho=1}^{k}\left\{\sum_{\zeta=1}^{r_\rho}\frac{\left[\sum_{\ell=s}^{r_\rho}\Delta_\ell^\rho\binom{m}{\ell-1}(-R_\rho)^{m+1-\ell}\right]}{(x+R_\rho)^\zeta}\right\} \tag{34}$$

Thus,

$$\begin{aligned} F(x) &= \frac{\sum_{m=0}^{n}N_m x^m}{(x+R_1)^{r_1}(x+R_2)^{r_2}\cdots(x+R_k)^{r_k}} \\ &= \sum_{m=0}^{n}N_m\left\{\sum_{\zeta=0}^{m-1}\left\{\sum_{\rho=1}^{k}\left[\sum_{\ell=1}^{m-\zeta}\Delta_\ell^\rho\binom{m-1-\zeta}{\ell-1}(-R_\rho)^{m-\zeta-\ell}\right]\right\}x^\zeta\right\} \\ &+\sum_{m=0}^{n}N_m\left\{\sum_{\rho=1}^{k}\left\{\sum_{\zeta=1}^{r_\rho}\frac{\left[\sum_{\ell=s}^{r_\rho}\Delta_\ell^\rho\binom{m}{\ell-1}(-R_\rho)^{m+1-\ell}\right]}{(x+R_\rho)^\zeta}\right\}\right\} \end{aligned} \tag{35}$$

and after expanding and regrouping we have

$$\begin{aligned} F(x) &= \sum_{\zeta=0}^{m-1}\left\{\sum_{m=0}^{n}N_m\left\{\sum_{\rho=1}^{k}\left[\sum_{\ell=1}^{m-\zeta}\Delta_\ell^\rho\binom{m-1-\zeta}{\ell-1}(-R_\rho)^{m-\zeta-\ell}\right]\right\}x^\zeta\right\} \\ &+\sum_{\rho=1}^{k}\left\{\sum_{\zeta=1}^{r_\rho}\frac{\sum_{m=1}^{n}N_m\left[\sum_{\ell=s}^{r_\rho}\Delta_\ell^\rho\binom{m}{\ell-1}(-R_\rho)^{m+1-\ell}\right]}{(x+R_\rho)^\zeta}\right\} \end{aligned} \tag{36}$$

4 Conclusion

(Insert conclusion)

5 References

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