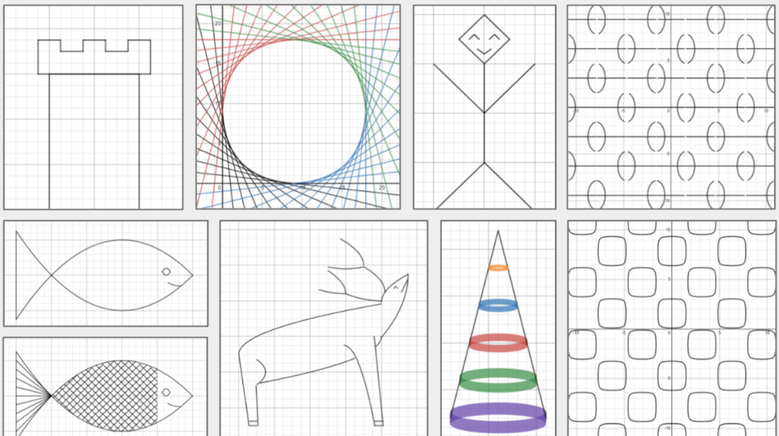
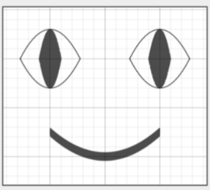
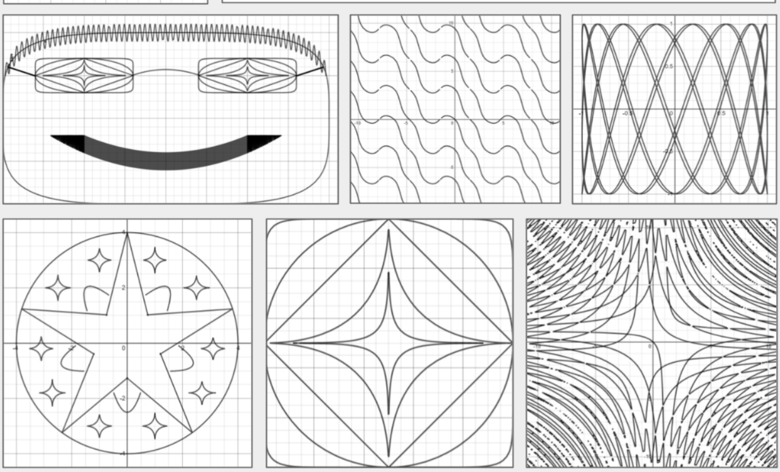




Graphing Calculator Drawing Exercises



Justin Skycak

www.justinmath.com

Introduction

During school I would sometimes pass time by drawing on my graphing calculator.

Years later in 2019, I turned this hobby into a summer course for the Math Academy program in the Pasadena Unified School District.

This workbook contains the lessons that were delivered during that course.

Familiarity with algebra is assumed.

Table of Contents

Chapter 1
Lines

1.1 Horizontal and Vertical Lines

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Vertical Lines. Observe the graph as you type each of the following inputs:

$$x = 1$$

$$x = 2$$

$$x = 3$$

Demonstration - Horizontal Lines. Observe the graph as you type each of the following inputs:

$$y = 1$$

$$y = 2$$

$$y = 3$$

Demonstration - Rays and Segments. Observe the graph as you type each of the following inputs:

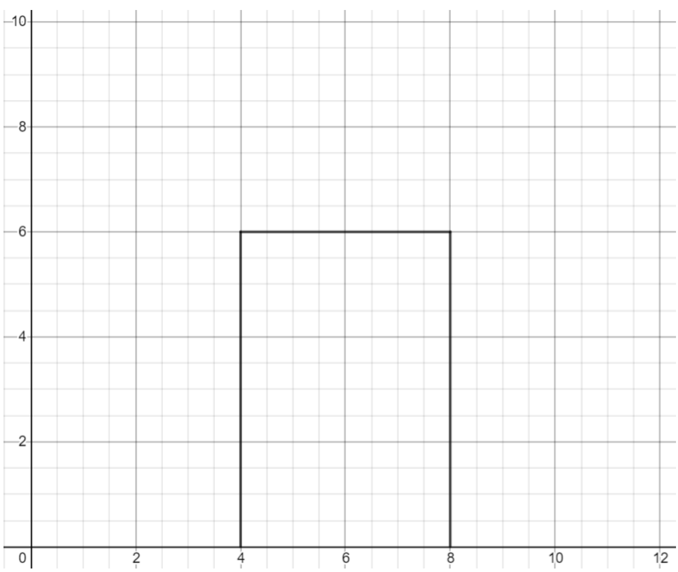
$$x = 1\{2 < y\}$$

$$x = 2\{2 < y < 3\}$$

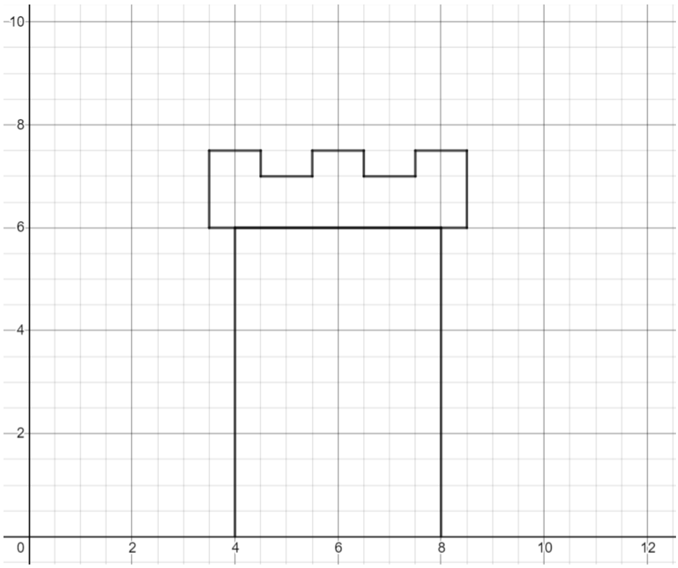
$$y = 1\{x < 2\}$$

$$y = 2\{0 < x < 2\}$$

Exercise. Reproduce the graph below. (Note: you can change the line color by clicking <u>and holding</u> the icon.)



Exercise. Add ridges to the top of the graph you produced in the previous example.



Challenge. Draw a full castle!

1.2 Slanted Lines

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Slope. Observe the graph as you type each of the following inputs. In general, the line $y = mx$ goes m units up per unit it goes right.

$$y = 10x$$

$$y = 1x$$

$$y = 0.1x$$

$$y = 0x$$

$$y = -0.1x$$

$$y = -1x$$

$$y = -10x$$

Demonstration - Intercept. Observe the graph as you type each of the following inputs. In general, the graph $y = mx + b$ crosses the y-axis at the point $(0, b)$.

$$y = x + 5$$

$$y = x + 2$$

$$y = x + 1$$

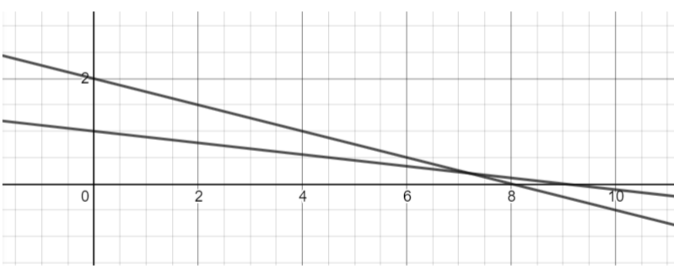
$$y = x + 0$$

$$y = x - 1$$

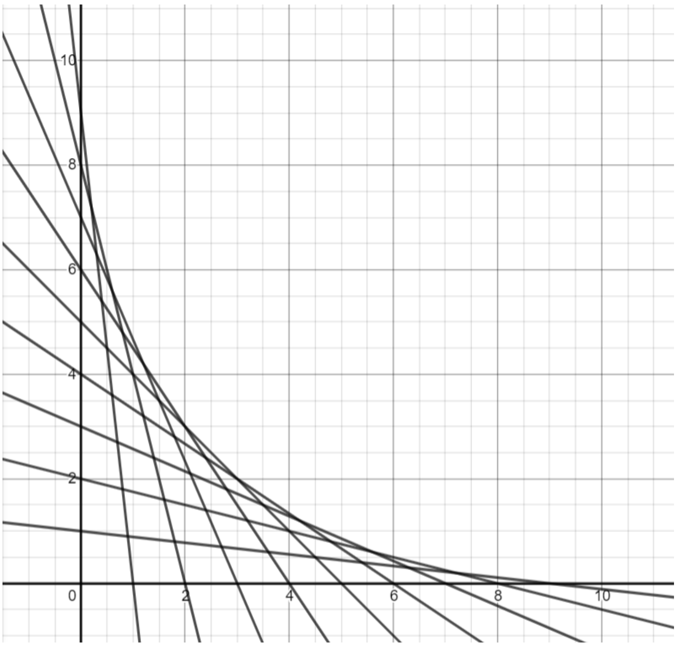
$$y = x - 2$$

$$y = x - 5$$

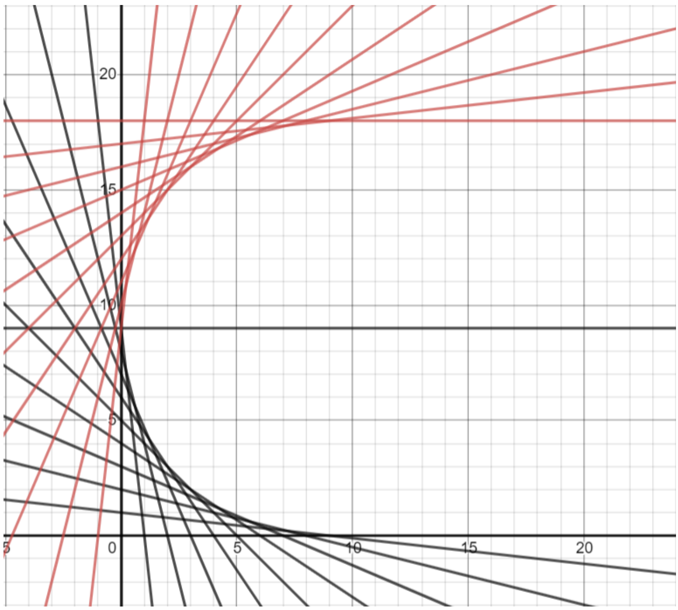
Exercise. Draw the two lines shown below. (Hint: one of the lines is given by $y = 1 - \frac{1}{9}x$)



Exercise. Draw more lines to reproduce the "spider web" graph shown below.



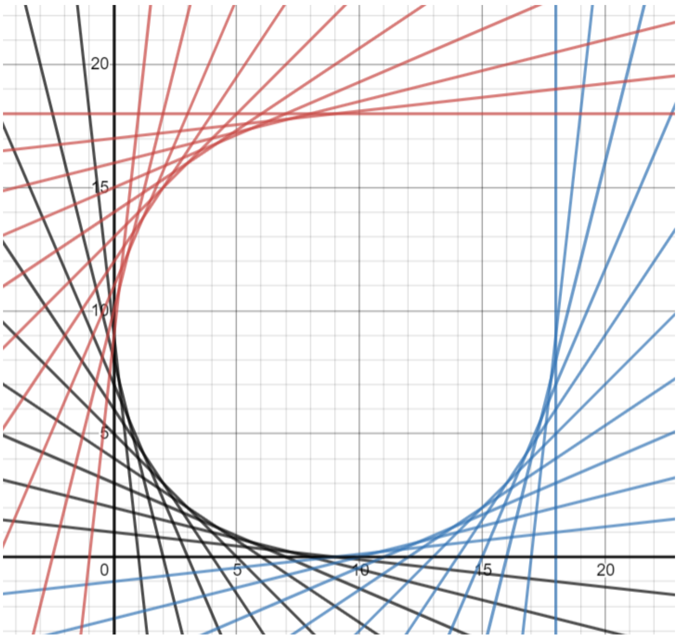
Exercise. Draw more lines to reflect the spider web upwards. (Hint: starting with the lines you drew previously, you can make the slopes positive, and adjust the intercepts as needed.)



Demonstration. The equation $y = m(x-a)+b$ creates a line with slope m through the point (a,b).

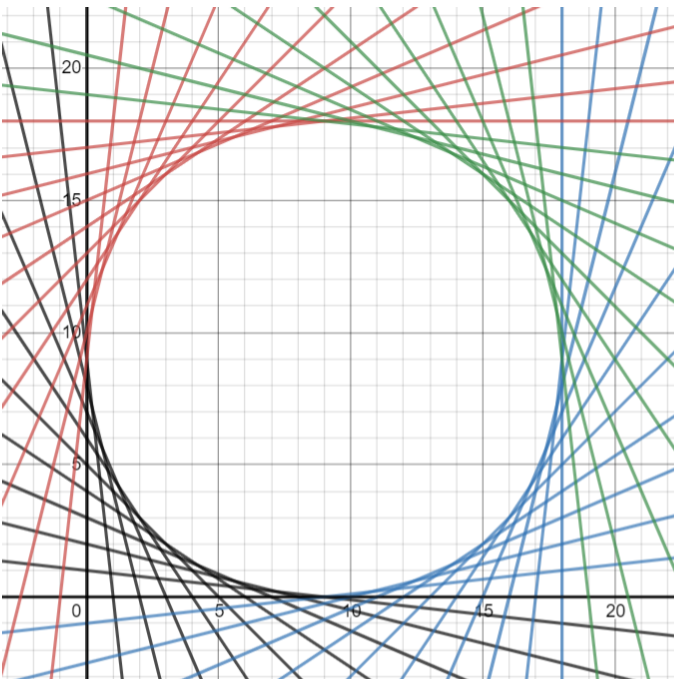
- The line through $(9,0)$ with slope $\frac{1}{9}$ is given by $y = \frac{1}{9}(x-9)+0$.
- The line through $(10,0)$ with slope $\frac{2}{8}$ is given by $y = \frac{2}{8}(x-9)+0$.

Exercise. Draw more lines to complete the bottom-right portion of your spider web. Two of the lines are given in the previous demonstration.



Exercise. Using the equation $y = m(x-a)+b$, complete the top-right corner of your spider web. Two lines are provided below.

- The line through $(18, 17)$ with slope $-\frac{1}{9}$ is given by $y = -\frac{1}{9}(x-18)+17$.
- The line through $(18, 16)$ with slope $-\frac{2}{8}$ is given by $y = -\frac{2}{8}(x-18)+16$.



1.3 Absolute Value

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Absolute Value. Observe the graph as you type each of the following inputs. In general, an absolute value graph $y = m|x|$ makes a "V" shape, with the magnitude of m controlling the slope of the V, and the sign of m controlling whether the V opens upward or downward.

$$y = 5|x|$$

$$y = 1|x|$$

$$y = 0.1|x|$$

$$y = -0.1|x|$$

$$y = -1|x|$$

$$y = -5|x|$$

Demonstration - Shifts. Observe the graph as you type each of the following inputs. In general, the graph of $y = m|x-a|+b$ shifts the absolute value graph $y=m|x|$ so that the pointy part of the “V” occurs at the point (a,b).

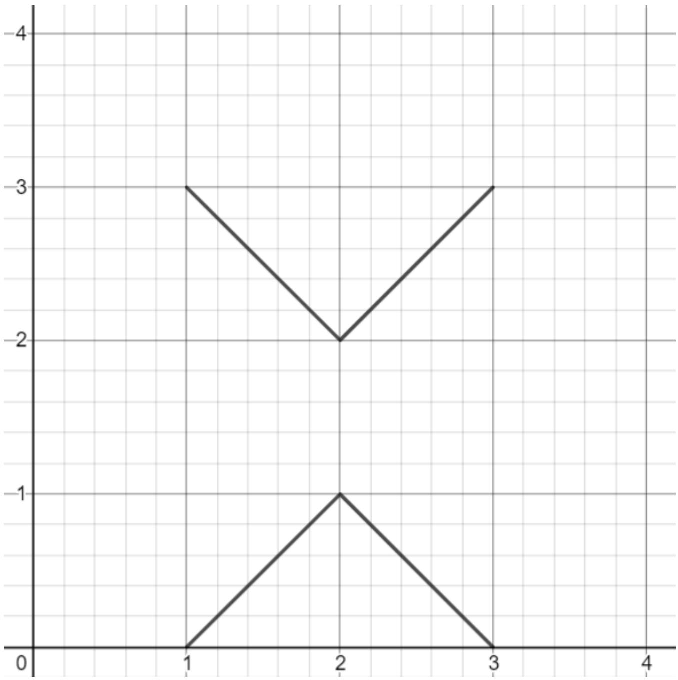
$$y = |x-1|+2$$

$$y = -2|x-1|-3$$

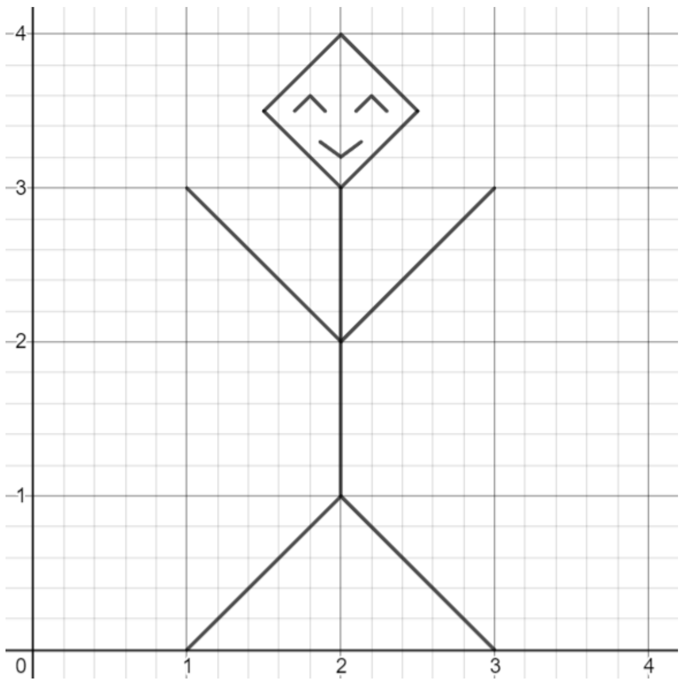
$$y = -0.5|x+3|-1$$

$$y = 10|x+2|+1$$

Exercise. Draw the two absolute value functions shown below. (Hint: Remember that you can limit the domain and range of your functions with parentheses, e.g. $y = |x|\{-1 < x < 1\}$ or $y = |x|\{y < 3\}$)



Exercise. Draw more absolute value functions to create a person! (The person's back will be a vertical line, but everything else can be made out of absolute value functions.)



Challenge. Try to draw yourself, or your friend! You can include hair, shoes, ears, hands, clothes, etc.)

Chapter 2
Open Curves

2.1 Parabolas

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Parabola. Observe the graph as you type each of the following inputs. In general, an absolute value graph $y = mx^2$ makes a “U” shape, with the magnitude of m controlling the slope of the U, and the sign of m controlling whether the U opens upward or downward.

$$y = 5x^2$$

$$y = 1x^2$$

$$y = 0.1x^2$$

$$y = -0.1x^2$$

$$y = -1x^2$$

$$y = -5x^2$$

Demonstration - Shifts. Observe the graph as you type each of the following inputs. In general, the graph of $y = m(x-a)^2 + b$ shifts the absolute value graph $y = mx^2$ so that the hump of the “U” occurs at the point (a, b).

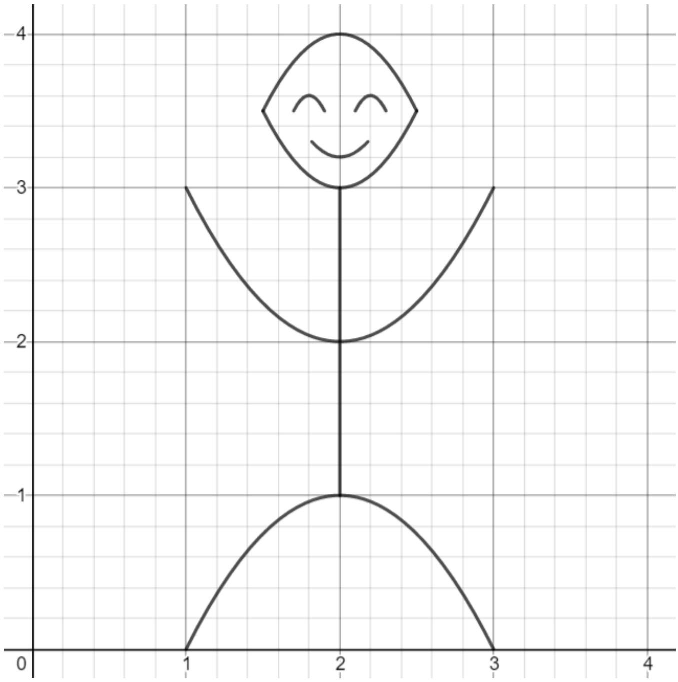
$$y = (x-1)^2 + 2$$

$$y = -2(x-1)^2 - 3$$

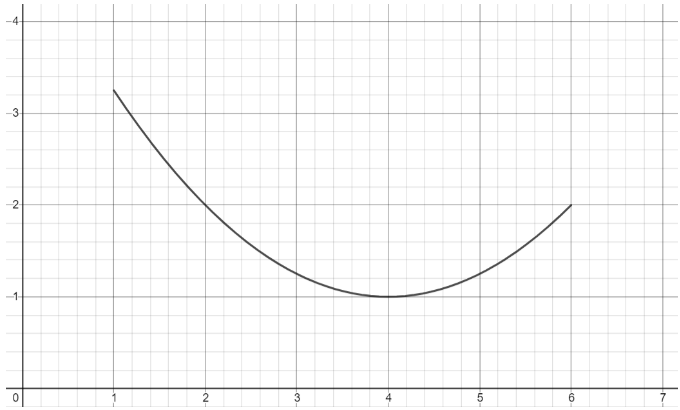
$$y = -0.5(x+3)^2 - 1$$

$$y = 10(x+2)^2 + 1$$

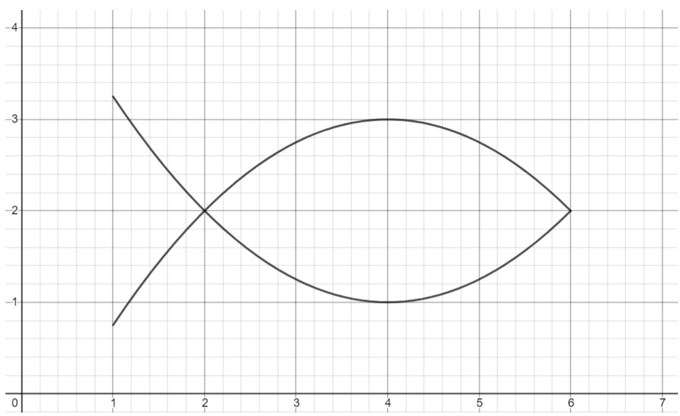
Exercise. Previously, you used absolute value functions to create a person. This time, draw the person using parabolas!



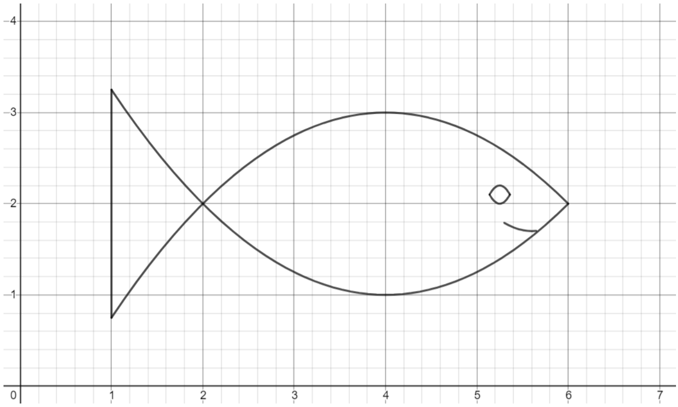
Exercise. Draw the portion of the parabola shown below.



Exercise. Reflect and shift a copy of the parabola portion drawn previously to form an outline of a fish.



Exercise. Complete the final details of the fish.



Challenge. Draw a school of fish! You can try to include other sea creatures, as well.

2.2 Sine Waves

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Equilibrium. Observe the graph as you type each of the following inputs. In general, the graph of $y = \sin x$ looks like an infinite wavy squiggle oscillating up and down around an equilibrium at $y = 0$. The graph $y = \sin(x) + b$ shifts the equilibrium of the wavy squiggle to the line $y = b$.

$$y = \sin(x) + 5$$

$$y = \sin(x) + 1$$

$$y = \sin(x) + 0$$

$$y = \sin(x) - 1$$

$$y = \sin(x) - 5$$

Demonstration - Frequency. Observe the graph as you type each of the following inputs. The “frequency” of a sine wave refers to how quickly or “frequently” it oscillates. For a sine wave $y = \sin(vx)$, the frequency is controlled by v. If you double v, then the sine wave will oscillate twice as frequently; if you halve v, then the sine wave will oscillate half as frequently. If you set $v = 0$, then the sine wave will not oscillate at all.

$$y = \sin(x)$$

$$y = \sin(2x)$$

$$y = \sin(4x)$$

$$y = \sin(x)$$

$$y = \sin(0.5x)$$

$$y = \sin(0.25x)$$

Demonstration - Amplitude. Observe the graph as you type each of the following inputs. The "amplitude" of a sine wave refers to how high/low its peaks/valleys are in relation to its equilibrium. For a sine wave $y = A\sin(x)$, the amplitude is controlled by A. The peaks of the sine wave reach a height of A, and the valleys of the sine wave reach a depth of $-A$.

$$y = 5\sin(x)$$

$$y = 1\sin(x)$$

$$y = 0.25\sin(x)$$

Demonstration - Horizontal Shift. Observe the graph as you type each of the following inputs. The sine graph $y = \sin(x - a)$ is shifted right a units, meaning that each peak and each valley occurs a units right of its original location.

$$y = \sin(x)$$

$$y = \sin(x - 1)$$

$$y = \sin(x - 1.57)$$

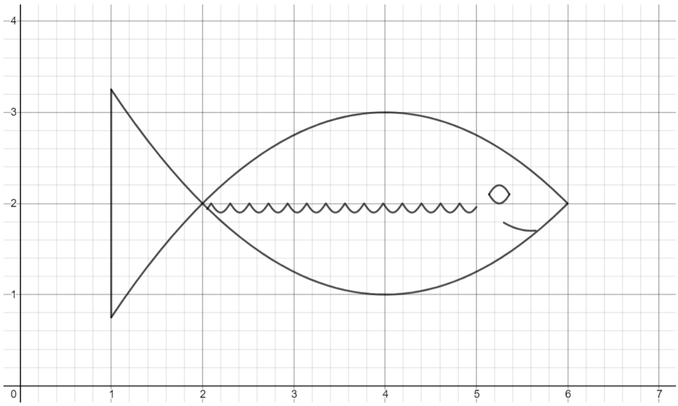
$$y = \sin(x - 2)$$

Demonstration - Composition with Absolute Value. Observe the graph as you type each of the following inputs.

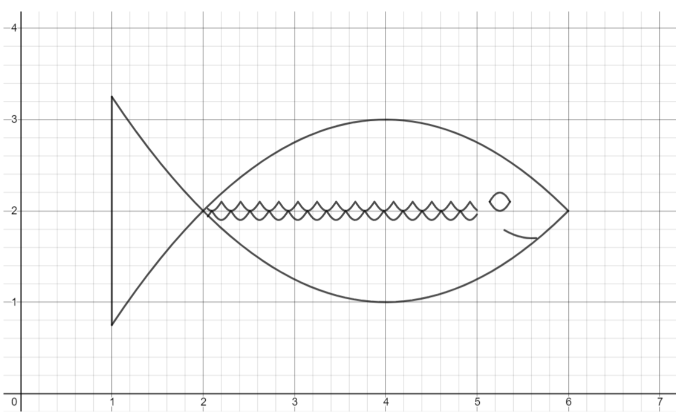
$$y = |\sin(x)|$$

$$y = -|\sin(x)|$$

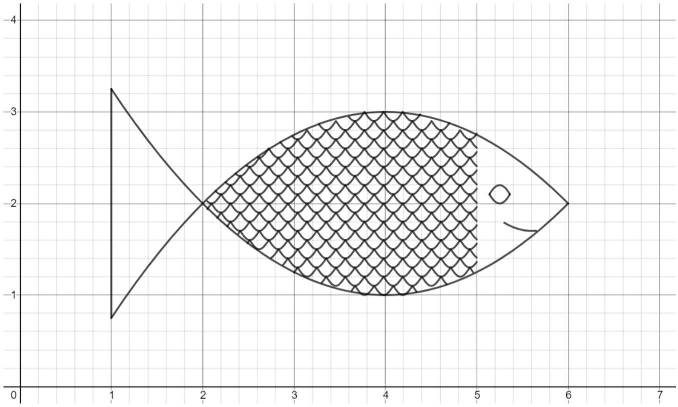
Exercise. Previously, you drew a fish using parabolas. Now, create a layer of scales on it, using a function of the form $y = -A|\sin(x)| + b$.



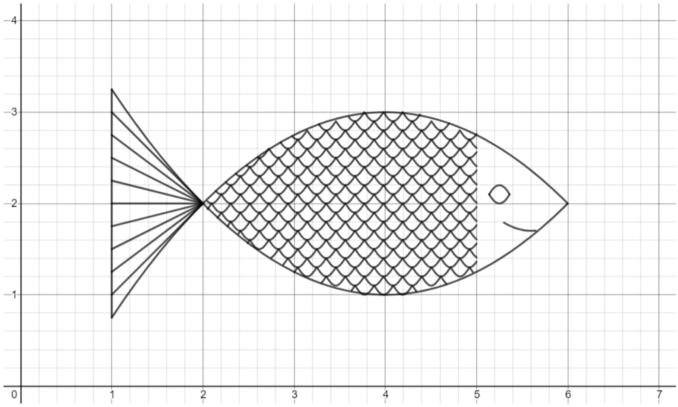
Exercise. Now, create a second layer of scales, using a function of the form $y = -A|\sin(x-a)| + b$. The peaks of the first layer should line up with the valleys of the second layer.



Exercise. Continue making layers of scales until the fish is completely scaled.



Exercise. Lastly, use lines to create spines in the tail of the fish.



Challenge. Try to draw other scaled creatures, such as a snake!

2.3 Roots

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Root Graph. Observe the graph as you type each of the following inputs. In general, the graph $y = m\sqrt{x-a}$ makes an "r" shape that is shifted right a units and stretched vertically by a factor of m.

$$y = \sqrt{x}$$

$$y = 5\sqrt{x}$$

$$y = 0.2\sqrt{x}$$

$$y = \sqrt{x-2}$$

$$y = \sqrt{x+3}$$

Demonstration - Reflections. Observe the graph as you type each of the following inputs. In general, the graph $y = \sqrt{-(x-a)}$ is the graph of $y = \sqrt{x-a}$ reflected over the y-axis, and the graph of $y = -\sqrt{x-a}$ is the graph of $y = \sqrt{x-a}$ reflected over the x-axis.

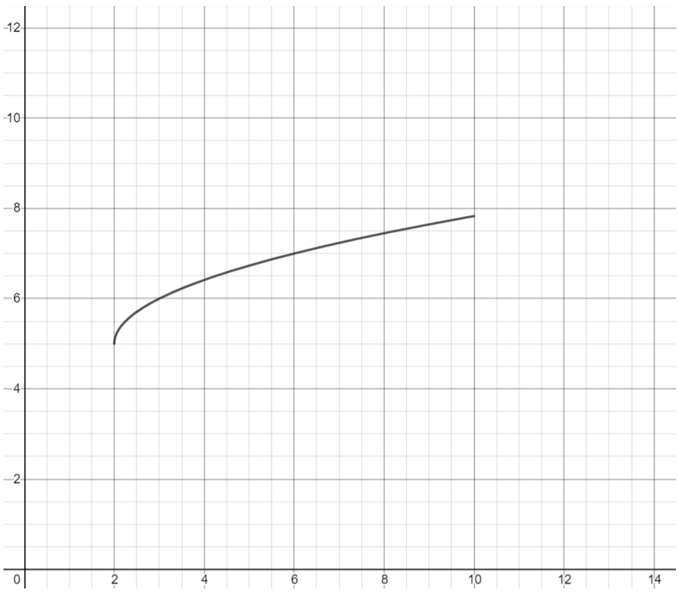
$$y = \sqrt{-x}$$

$$y = \sqrt{-(x-2)}$$

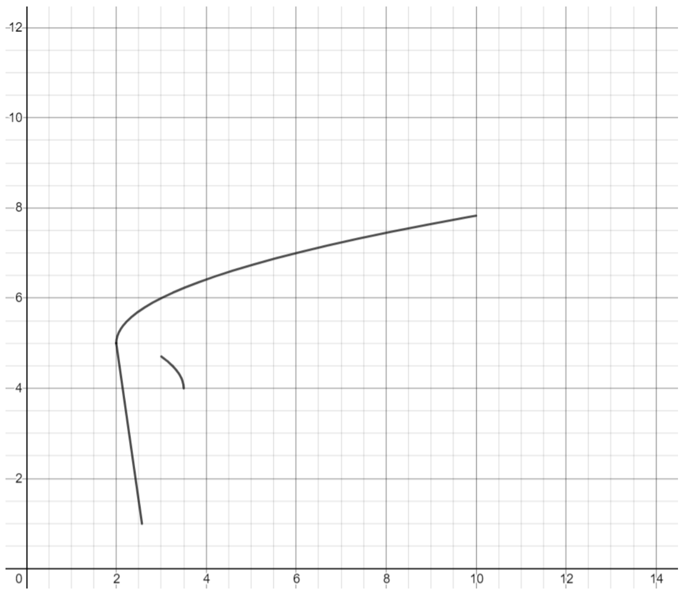
$$y = -\sqrt{x-5}$$

$$y = -\sqrt{-(x-5)}$$

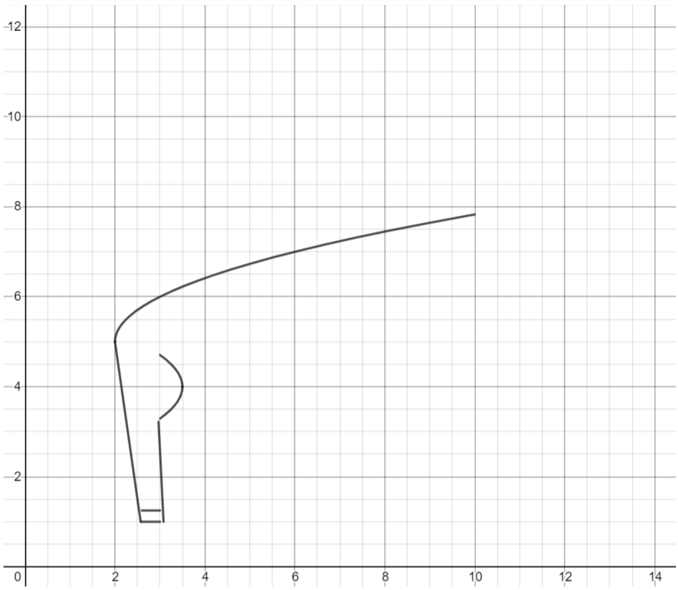
Exercise. Reproduce the graph shown below using a square root function of the form $y = m\sqrt{x-a}+b$. This will form the back of a “root deer”.



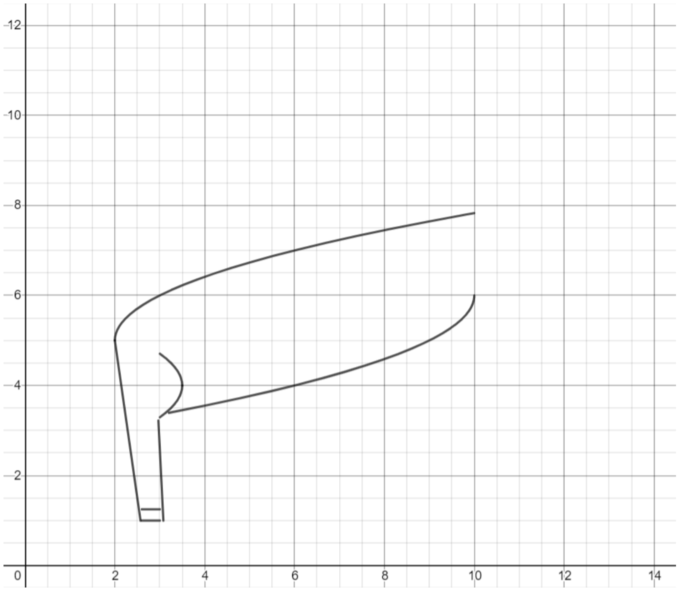
Exercise. Draw the back hip of the deer using a root function, and the back leg of the deer using a line.



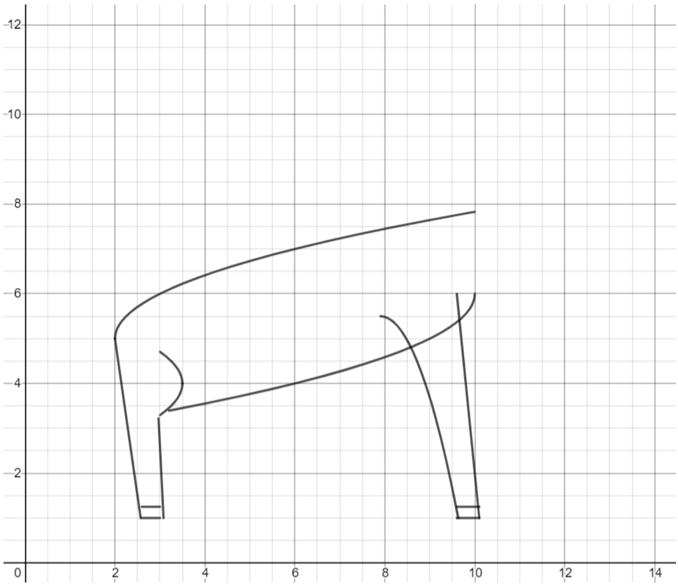
Exercise. Complete the back leg of the deer.



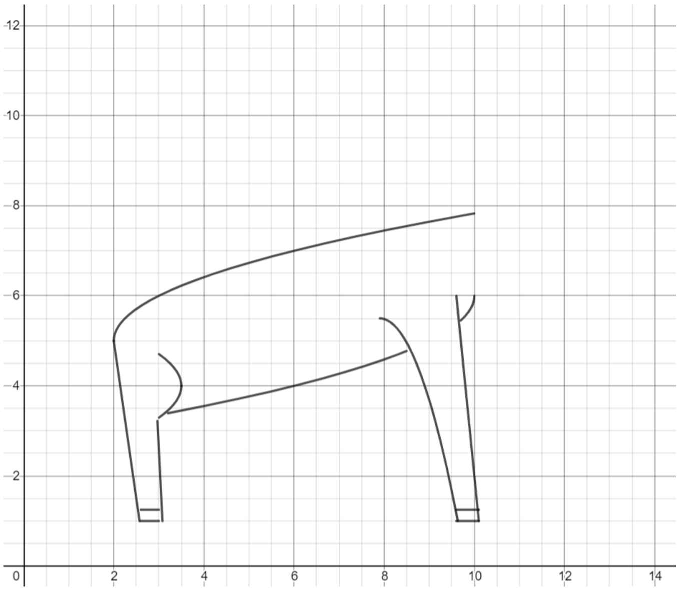
Exercise. Use another root function to draw the belly of the deer.



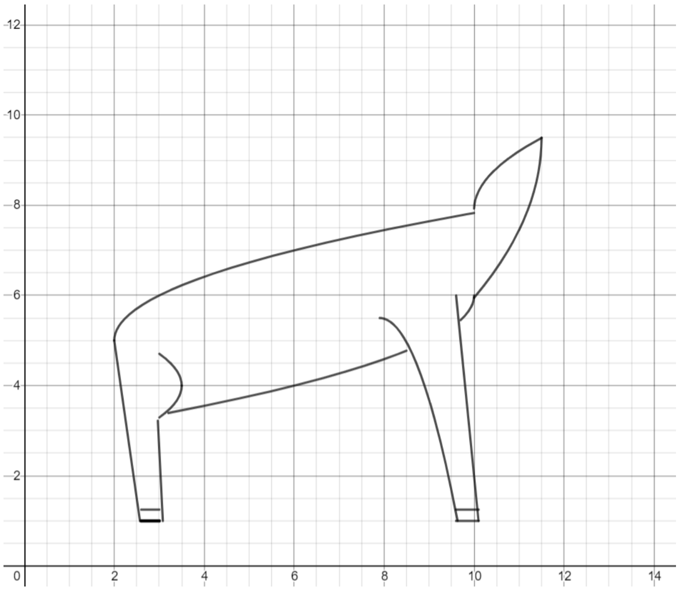
Exercise. Use lines and a parabola to draw the front leg of the deer.



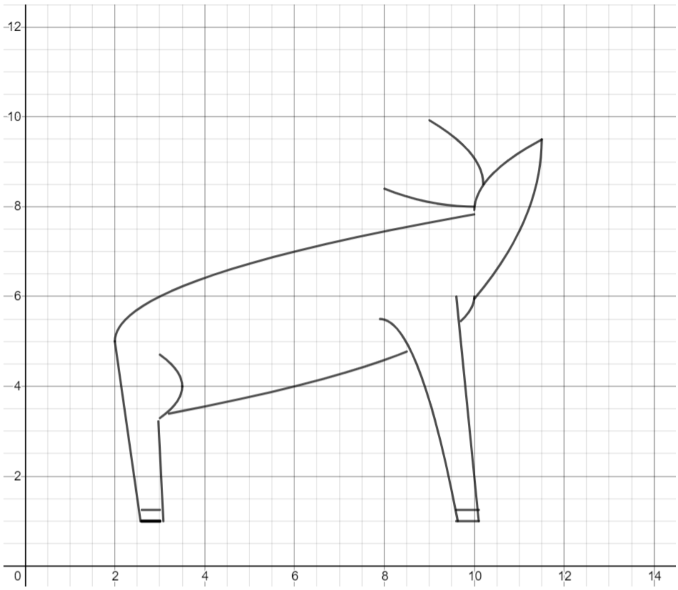
Exercise. Break the belly of the deer into two separate functions so that it does not cross over the deer's front leg.



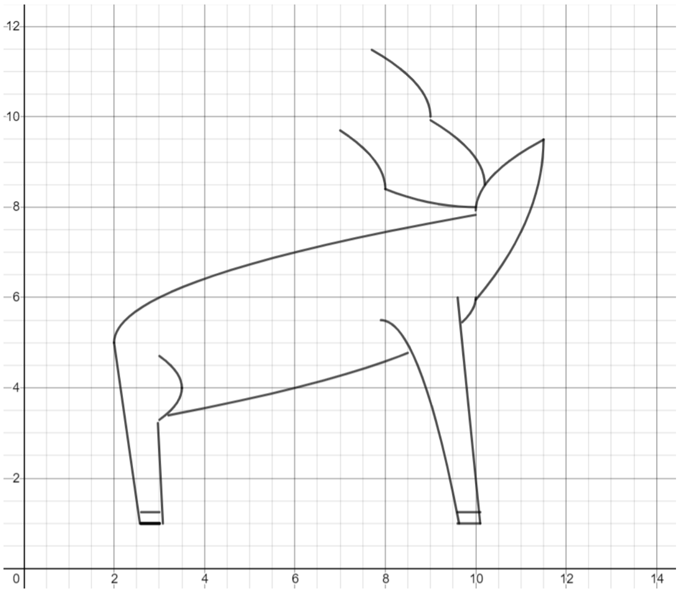
Exercise. Draw the head of the deer using two root functions.



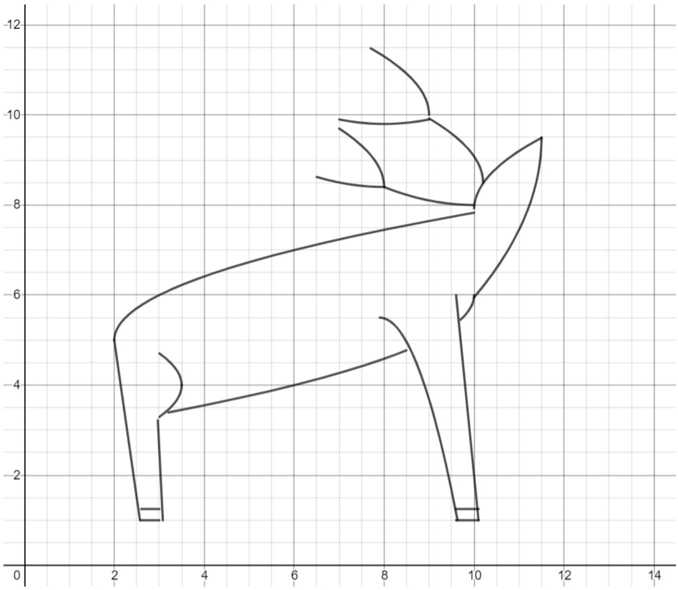
Exercise. Draw antlers on the deer. You can use a root function for the upper antler, and a parabola for the lower antler.



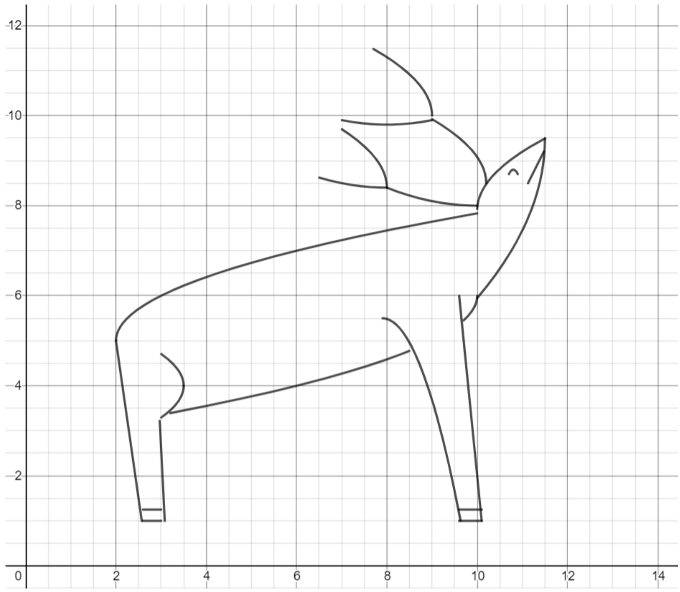
Exercise. Use a root function to draw an upper branch of antlers on each of the antlers previously drawn.



Exercise. Use a parabola to draw a lower branch of antlers below each of the antlers previously drawn.



Exercise. Use lines and a parabola to complete the face of the deer.



Challenge. Draw another woodland creature, such as a hedgehog.

Chapter 3
Closed Curves

3.1 Shading with Sine

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - High Frequency. Observe the graph as you type each of the following inputs. In general, the graph of $y = \sin(vx)$ looks more and more solid as v increases.

$$y = \sin(x)$$

$$y = \sin(10x)$$

$$y = \sin(100x)$$

$$y = \sin(1000x)$$

Demonstration - Thickening a Curve. Observe the graph as you type each of the following inputs. In general, for large v, the graph of $y = f(x) + A\sin(vx)$ thickens the curve $y = f(x)$ to have a vertical thickness of $2A$.

$$y = x + 0.1\sin(1000x)$$

$$y = x^2 + 0.1\sin(1000x)$$

$$y = x^2 + 0.5\sin(1000x)$$

$$y = \sin(x) + 0.1\sin(1000x)$$

$$y = \sin(x) + 0.5\sin(1000x)$$

Demonstration - Varying Amplitude. Observe the graph as you type each of the following inputs. In general, when v is large, the graph of $y = f(x)\sin(vx)$ shades the area between the graphs of $y = -f(x)$ and $y = f(x)$

$$y = x\sin(1000x)$$

$$y = x^2\sin(1000x)$$

$$y = (\sin x)\sin(1000x)$$

Demonstration - Shifts. Observe the graph as you type each of the following inputs. In general, the graph of $y = f(x-a)\sin(vx) + b$ is the graph of $y = f(x)\sin(vx)$ shifted right by a units and up by b units.

$$y = x^2\sin(1000x)$$

$$y = (x-2)\sin(1000x) + 5$$

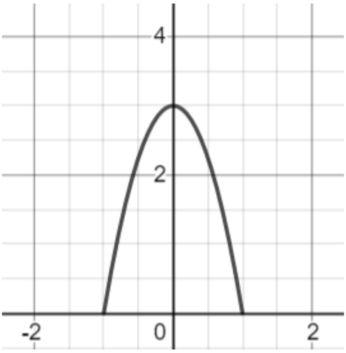
Demonstration - Limitations. Observe the graph as you type each of the following inputs.

$$y = \sin(1000x)\{5 < x < 10\}$$

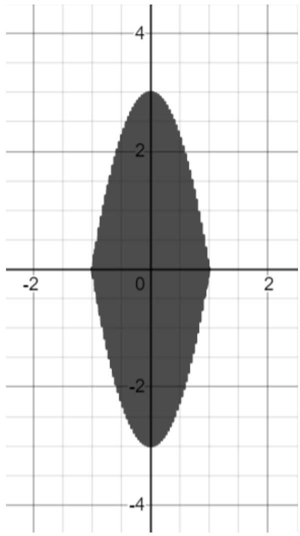
$$y = (x-2)^2 \sin(1000x)\{2 < x < 4\}$$

$$y = \left(\sqrt{x-3}\right) \sin(1000x) + 5\,\{3 < x < 4\}$$

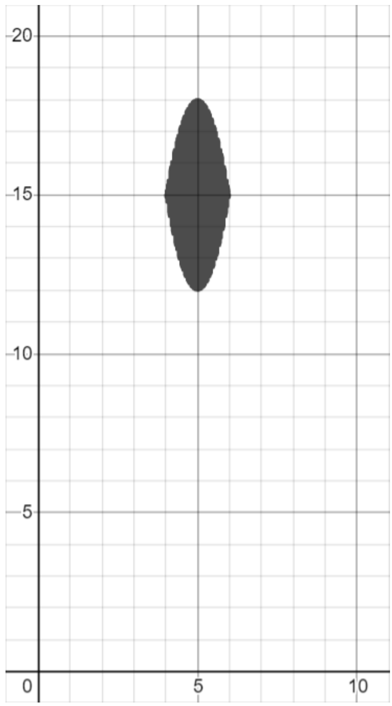
Exercise. Reproduce the downward parabola shown below.



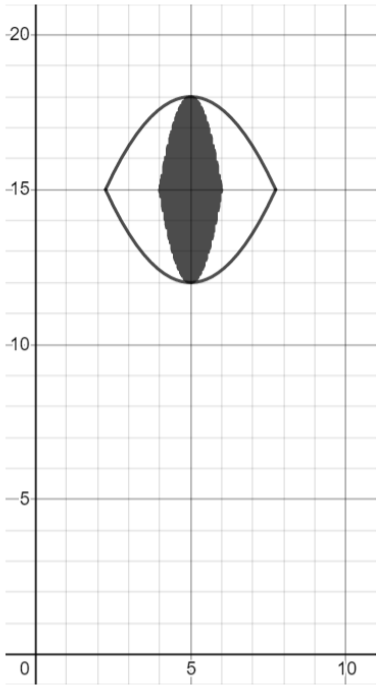
Exercise. Use the parabola as the amplitude of a high-frequency sine function to create a shaded area.



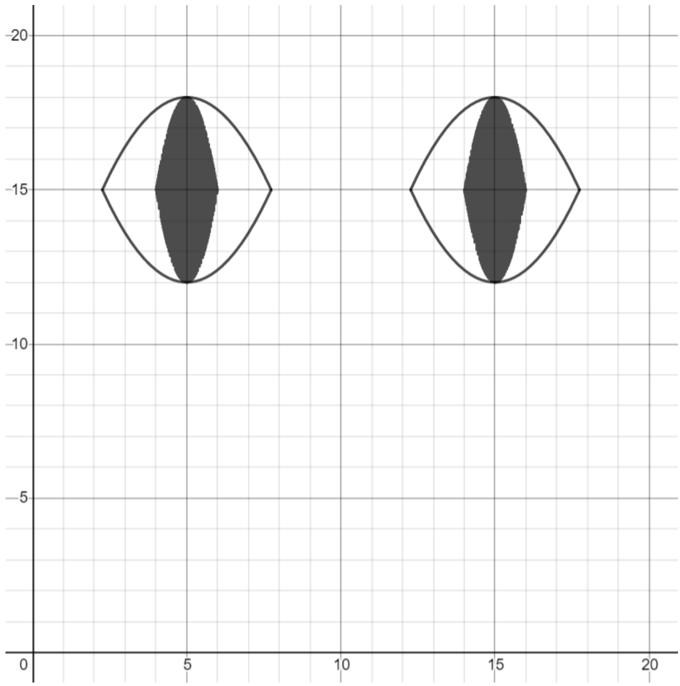
Exercise. Shift the shaded area up and right.



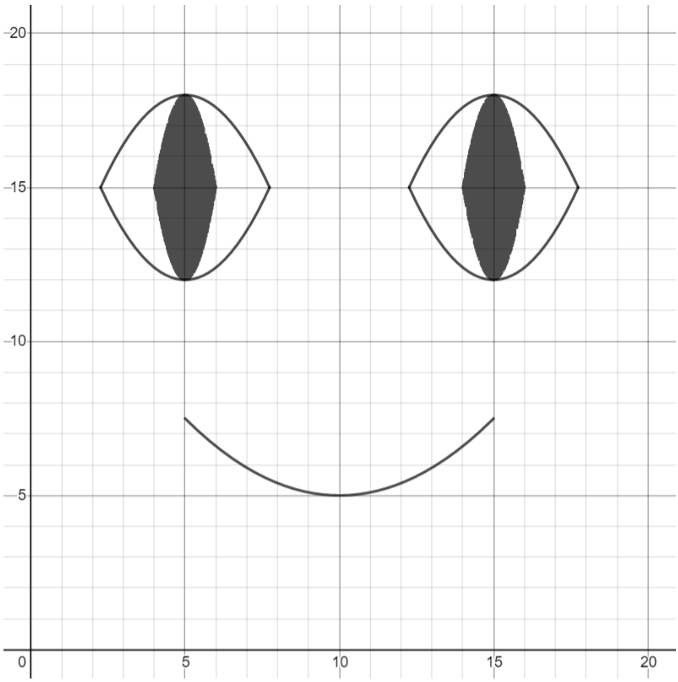
Exercise. Draw parabolas around the shaded area to create an eye.



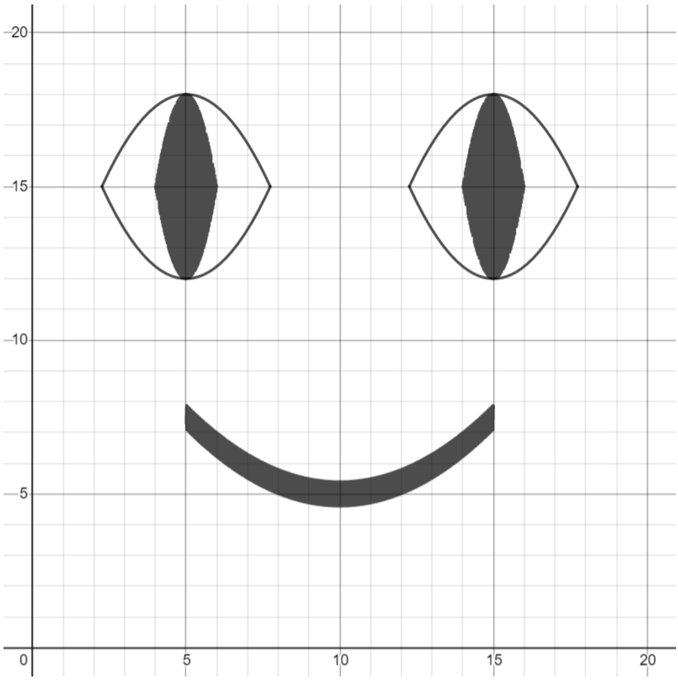
Exercise. Create another eye.



Exercise. Create a parabola in the shape of a mouth.



Exercise. Thicken the parabola that forms the shape of the mouth.



Challenge. Make other kinds of emoji faces, such as a sad face or a laughing face.

3.2 Euclidean Ellipses

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Circles. Observe the graph as you type each of the following inputs. In general, the graph of $(x-a)^2+(y-b)^2=r^2$ makes a circle with radius r centered at the point (a,b)

$$(x-0)^2+(y-0)^2=1^2$$

$$(x-2)^2+(y-3)^2=2^2$$

$$(x+5)^2+(y-2)^2=3^2$$

Demonstration - Ellipses. Observe the graph as you type each of the following inputs. In general, the graph of $\left(\frac{x-a}{A}\right)^2+\left(\frac{y-b}{B}\right)^2=1$ makes an ellipse with horizontal radius A and vertical radius B centered at the point (a,b).

$$\left(\frac{x-0}{1}\right)^2+\left(\frac{y-0}{2}\right)^2=1$$

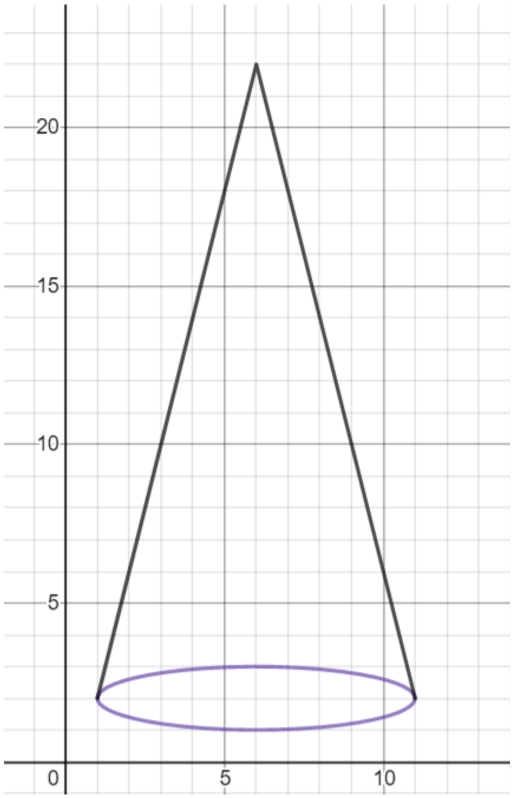
$$\left(\frac{x-2}{2}\right)^2+\left(\frac{y-3}{5}\right)^2=1$$

Demonstration - Shading. Observe the graph as you type each of the following inputs. In general, for large v, the graph of $\left(\frac{x-a}{A}\right)^2 + \left(\frac{y+A\sin(vx)-b}{B}\right)^2 = 1$ shades vertically around the graph of $\left(\frac{x-a}{A}\right)^2 + \left(\frac{y-b}{B}\right)^2 = 1$ with thickness $2A$.

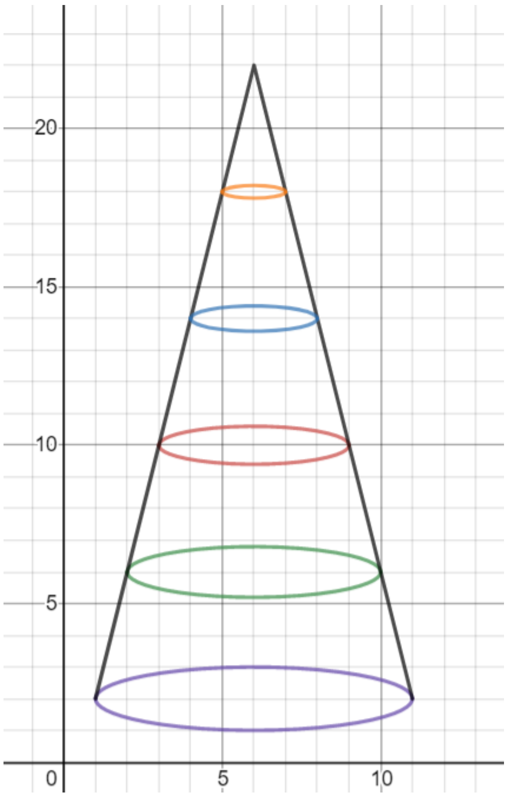
$$\left(\frac{x-0}{1}\right)^2 + \left(\frac{y+.1\sin(1000x)-0}{2}\right)^2 = 1$$

$$\left(\frac{x-2}{2}\right)^2 + \left(\frac{y+.4\sin(1000x)-3}{5}\right)^2 = 1$$

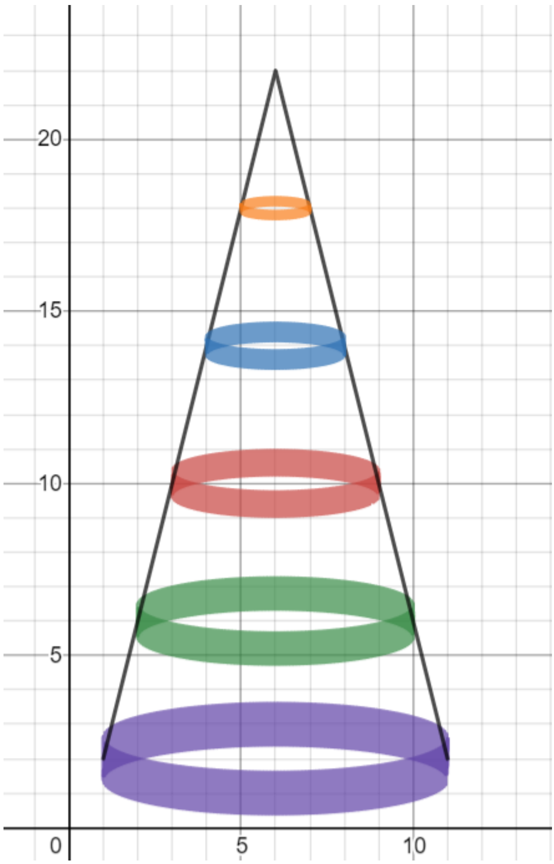
Exercise. Use an absolute value function together with an ellipse to draw a cone.



Exercise. Stack ellipses vertically on the cone.



Exercise. Thicken the ellipses to form cylindrical shells.



Challenge. Try stacking cylindrical shells on the peaks of $y = \sin x$.

3.3 Non-Euclidean Ellipses

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Non-Euclidean Circles. Observe the graph as you type each of the following inputs. In general, the graph of the unit circle is given by $|x|^n + |y|^n = 1$. For $n = 2$, this makes a Euclidean circle, i.e. all those points whose distance from the origin is 1, where distance is measured by the Euclidean metric $\sqrt{x^2 + y^2}$. For other values of n, these equations make non-Euclidean circles, i.e. all the points whose "distance" from the origin is 1, where distance is measured by the metric $\sqrt[n]{|x|^n + |y|^n}$.

$$|x|^4 + |y|^4 = 1$$

$$|x|^3 + |y|^3 = 1$$

$$|x|^2 + |y|^2 = 1$$

$$|x|^1 + |y|^1 = 1$$

$$|x|^{0.8} + |y|^{0.8} = 1$$

$$|x|^{0.5} + |y|^{0.5} = 1$$

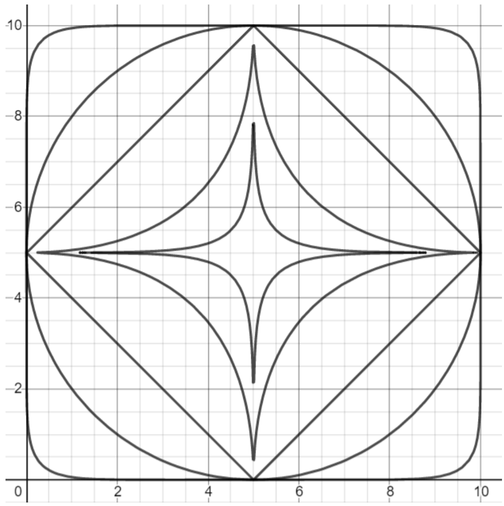
$$|x|^{0.2} + |y|^{0.2} = 1$$

Demonstration - Non-Euclidean Ellipses. Observe the graph as you type each of the following inputs. In general, the graph of $\left|\frac{x-a}{A}\right|^n + \left|\frac{y-b}{B}\right|^n = 1$ makes an ellipse with horizontal radius A and vertical radius B centered at the point (a,b). When $n \neq 2$, this is a non-Euclidean ellipse.

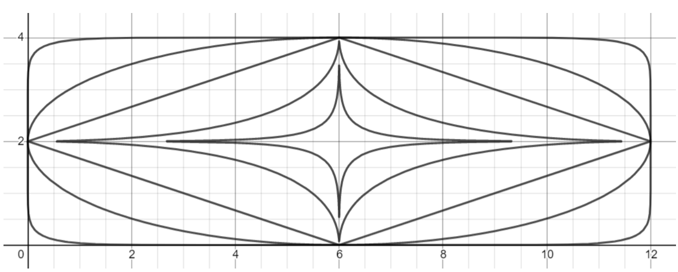
$$\left|\frac{x-0}{1}\right|^{10} + \left|\frac{y-0}{2}\right|^{10} = 1$$

$$\left|\frac{x-3}{2}\right|^{0.8} + \left|\frac{y-5}{5}\right|^{0.8} = 1$$

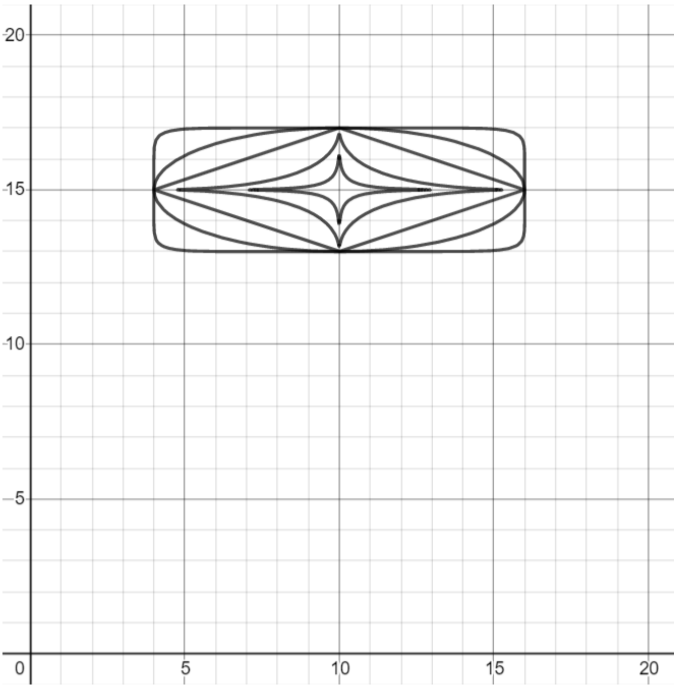
Exercise. Reproduce the graph shown below using non-Euclidean circles.



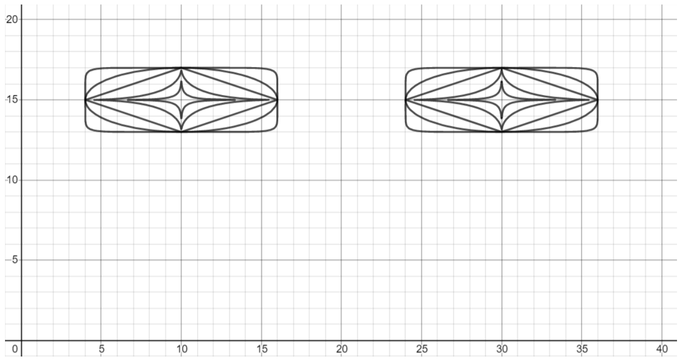
Exercise. Change the non-Euclidean circles to non-Euclidean ellipses in the previous exercise to reproduce the graph shown below.



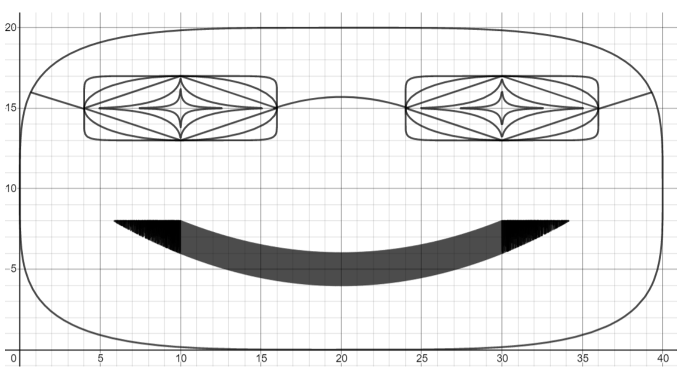
Exercise. Shift the ellipses right and up to produce the graph below.



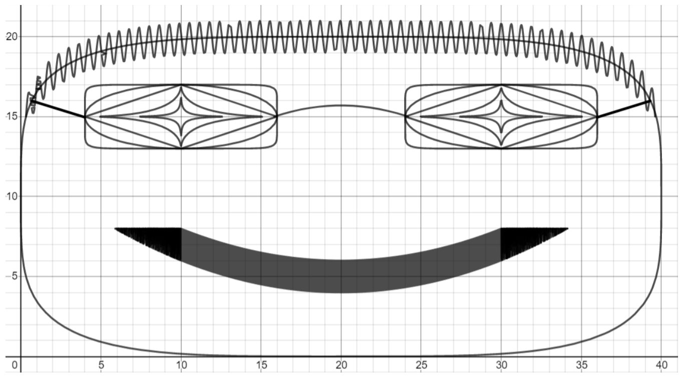
Exercise. Create another set of ellipses, shifted right of the original set.



Exercise. Add some details to form a face. The head can be made using a non-Euclidean ellipse, the frame of the glasses can be made using a parabola and two lines, and the smile can be made using a parabola with sine shading.



Exercise. Lastly, add some hair on the head. You can do this by duplicating the biggest ellipse that outlines the face, restricting the range, and shading via sine.



Challenge. Try to make a narrower face with longer hair.

Chapter 4
Trigonometry

4.1 Rotation

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Rotation. Observe the graph as you type each of the following inputs. In general, a graph can be rotated by an angle of θ about the origin by replacing x and y with the following expressions:

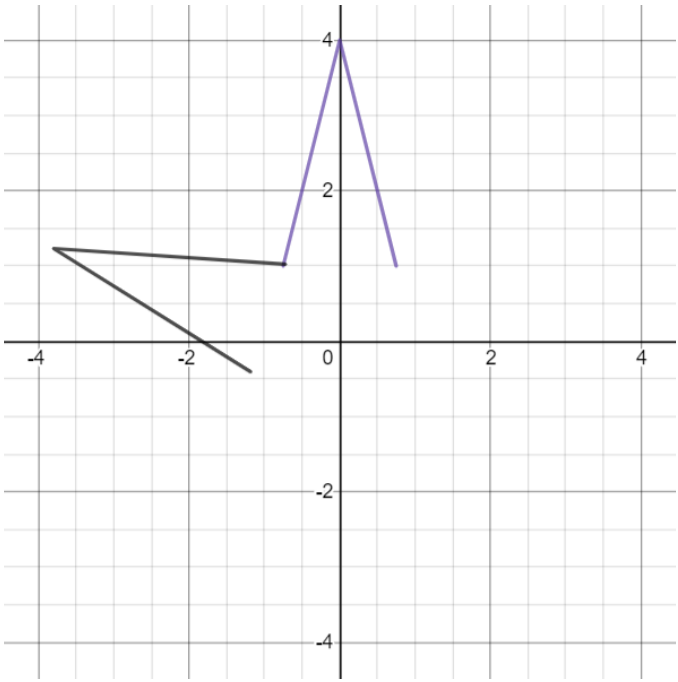
$$x \to x\cos\theta + y\sin\theta$$
$$y \to y\cos\theta - x\sin\theta$$

Note that θ should be given in radians, and one can convert degrees to radians by multiplying by the conversion factor $\frac{\pi}{180}$.

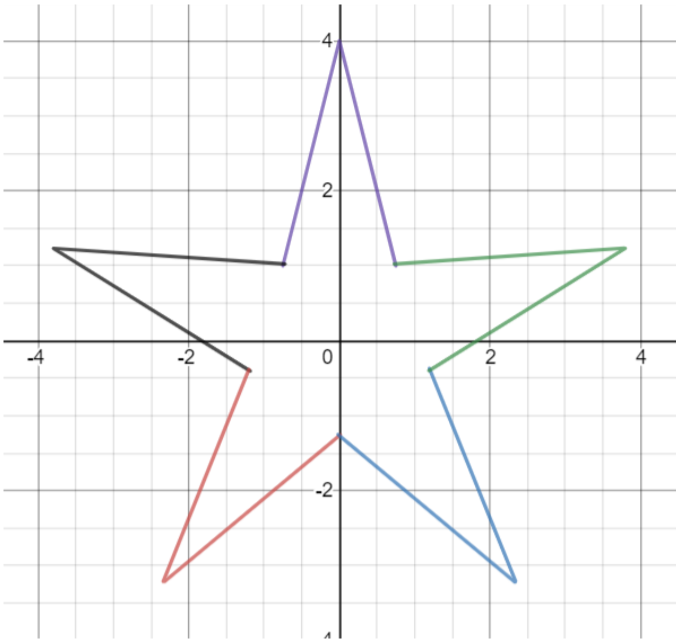
$$y\cos\tfrac{\pi}{6} - x\sin\tfrac{\pi}{6} = \left(x\cos\tfrac{\pi}{6} + y\sin\tfrac{\pi}{6}\right)^2$$

$$\left(\frac{x\cos\frac{\pi}{4}+y\sin\frac{\pi}{4}}{4}\right)^2 + \left(\frac{y\cos\frac{\pi}{4}-x\sin\frac{\pi}{4}}{2}\right)^2 = 1$$

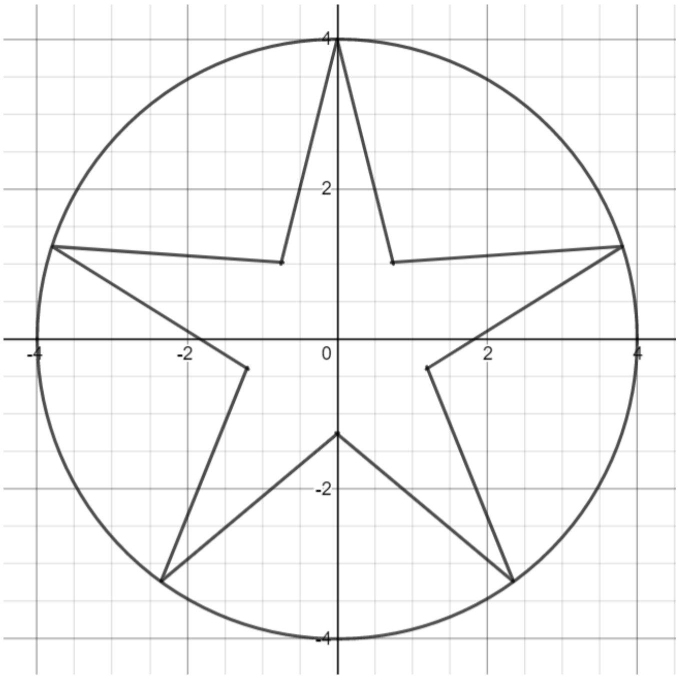
Exercise. Reproduce the graph below by drawing an absolute value function and then rotating it a fifth of a circle counterclockwise.



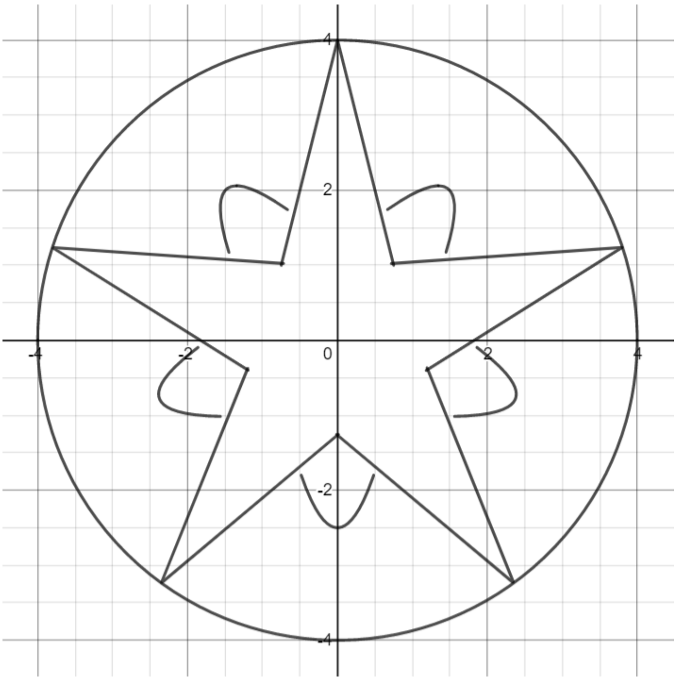
Exercise. Continue drawing rotated absolute value functions to form a star.



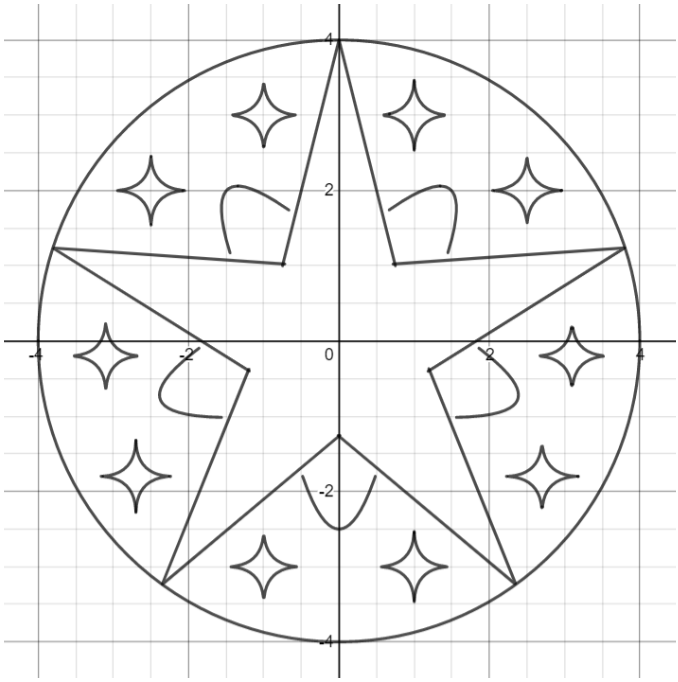
Exercise. Draw a circle that passes through the sharp points of the star.



Exercise. Add a background layer by drawing rotated parabolas.



Exercise. Finally, add non-Euclidean ellipses to the background.



Challenge. Create your own emblem.

4.2 Lissajous Curves

Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Lissajous Curves. Lissajous curves take the form

$$x = \sin(t)$$
$$y = \sin(at + b)$$

for some values of a and b. Observe the graph as you type each of the following Lissajous plot inputs, with $0 \leq t \leq 100$.

$$(\sin(t), \sin(t + 1))$$

$$(\sin(t), \sin(t + 2))$$

$$(\sin(t), \sin(t + 3))$$

$$(\sin(t), \sin(2t + 1))$$

$$(\sin(t), \sin(3t + 1))$$

$$(\sin(t), \sin(4t + 1))$$

$$(\sin(t), \sin(5t + 1))$$

$$(\sin(t), \sin(1.1t + 1))$$

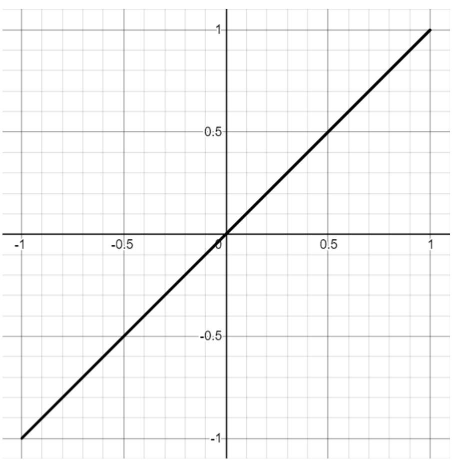
$$(\sin(t), \sin(1.2t+1))$$

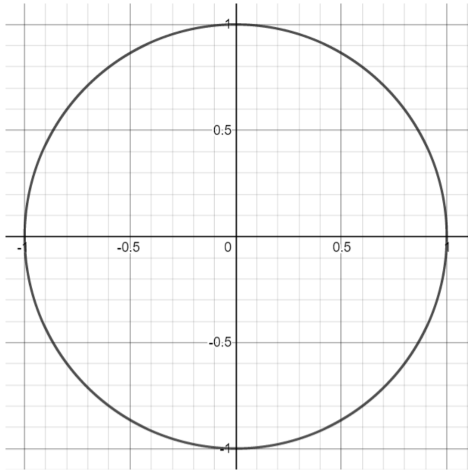
$$(\sin(t), \sin(1.3t+1))$$

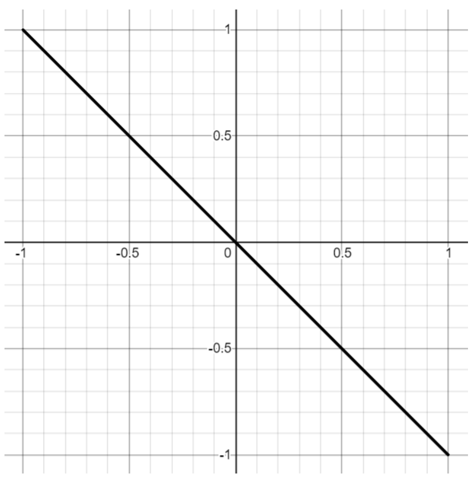
$$(\sin(t), \sin(1.4t+1))$$

$$(\sin(t), \sin(1.5t+1))$$

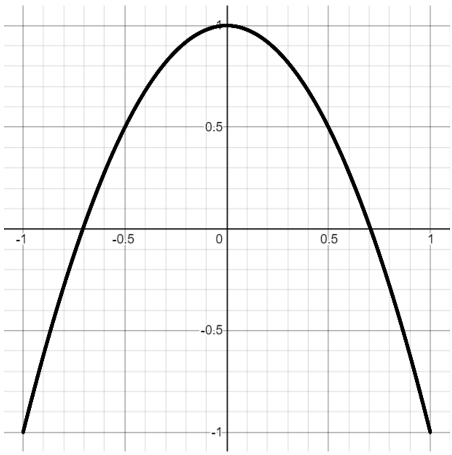
Exercise. Attempt to reproduce the graphs below by setting $a=1$ and varying the b parameter in the Lissajous curve equations. You may have to play with the parameter a bit to get a sense of what it controls.

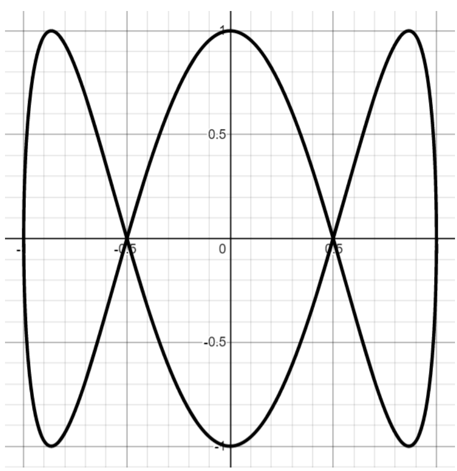


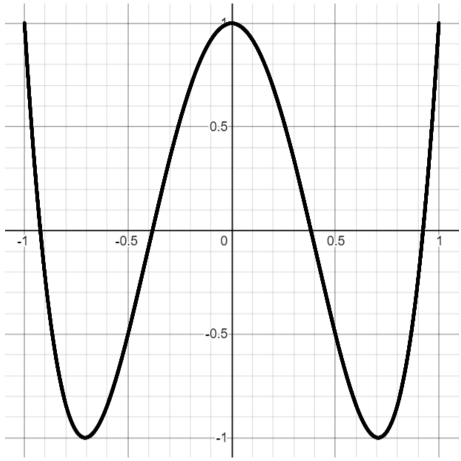

1
0.5
-1
-0.5
0
0.5
1
-0.5
-1

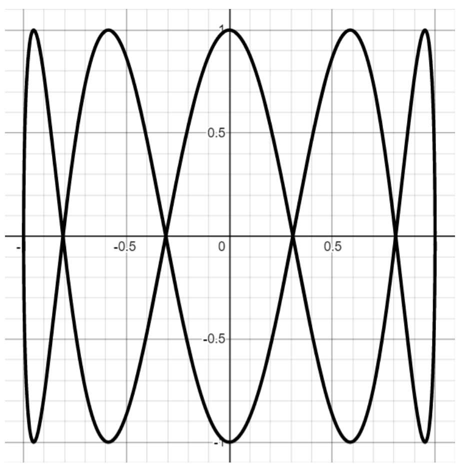

1
0.5
-1
-0.5
0
0.5
1
-0.5
-1

Exercise. Attempt to reproduce the graphs below by setting $b = \frac{\pi}{2}$ and varying the a parameter in the Lissajous curve equations. You may have to play with the parameter a bit to get a sense of what it controls.

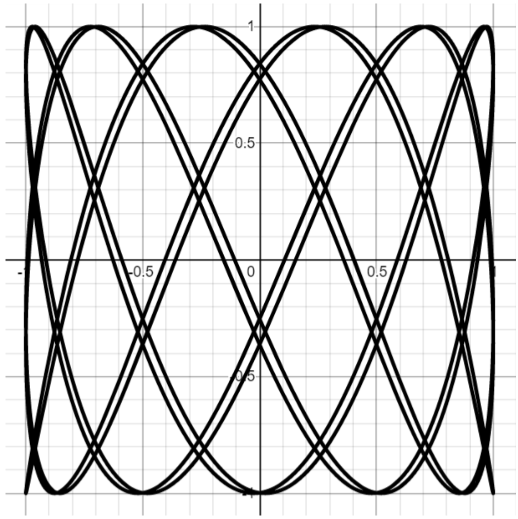


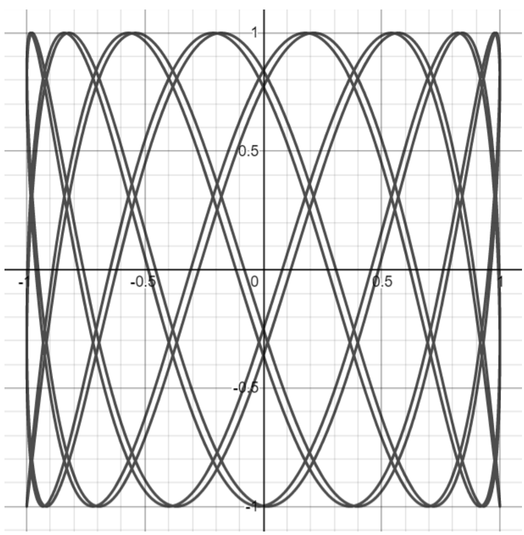


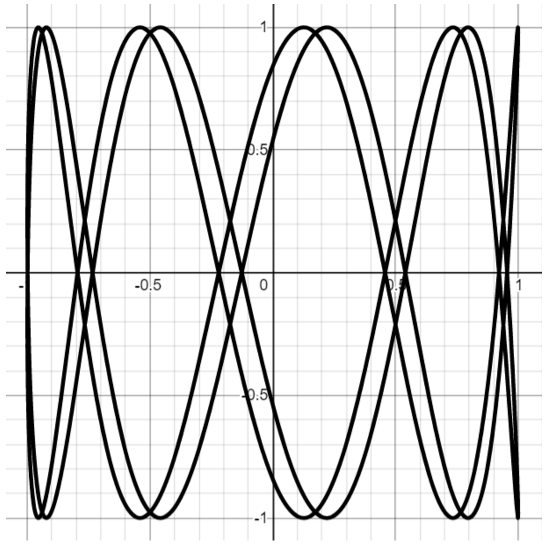


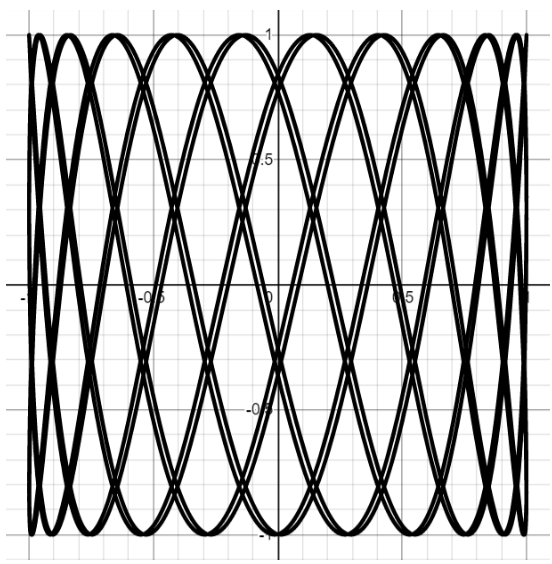


Challenge. Attempt to reproduce the Lissajous graphs below by setting $b = 1$ and varying a.


1
0.5
-1
-0.5
0
0.5
1
-0.5
-1


1
0.5
-1
-0.5
0
0.5
1
-0.5
-1



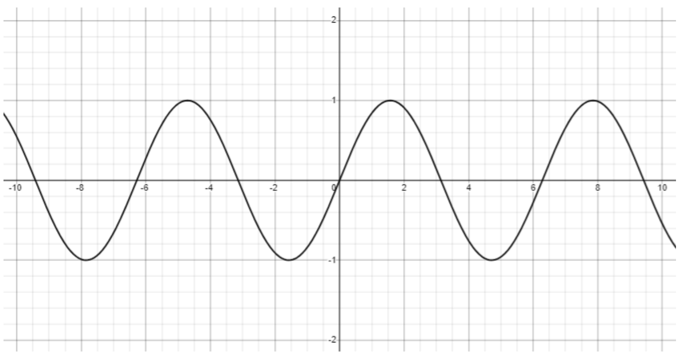


4.3 Composition Waves and Implicit Trig Patterns

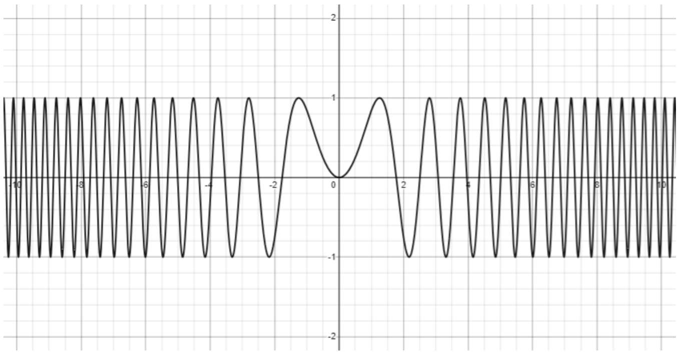
Setup. Navigate to https://www.desmos.com/calculator. Be sure to sign in so that you can save your graph.

Demonstration - Composition Waves. Observe the graph as you type each of the following inputs.

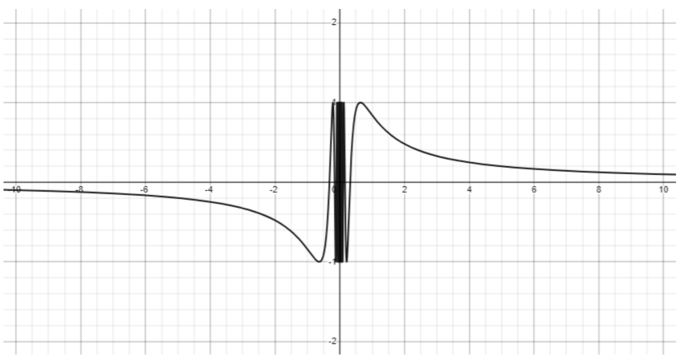
$$y = \sin(x)$$



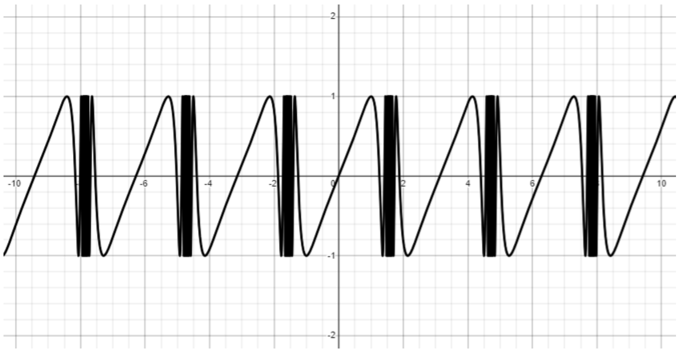
$$y = \sin(x^2)$$



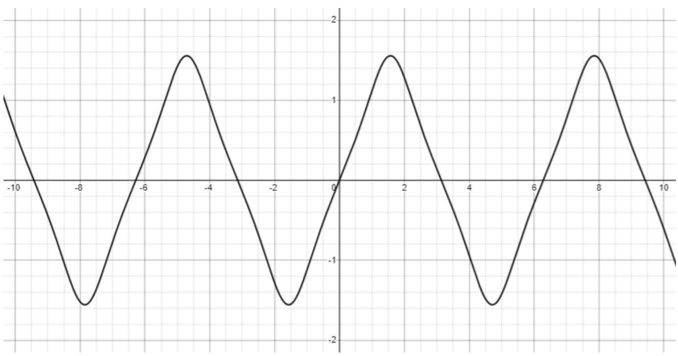
$$y = \sin\left(\frac{1}{x}\right)$$



$$y = \sin(\tan(x))$$

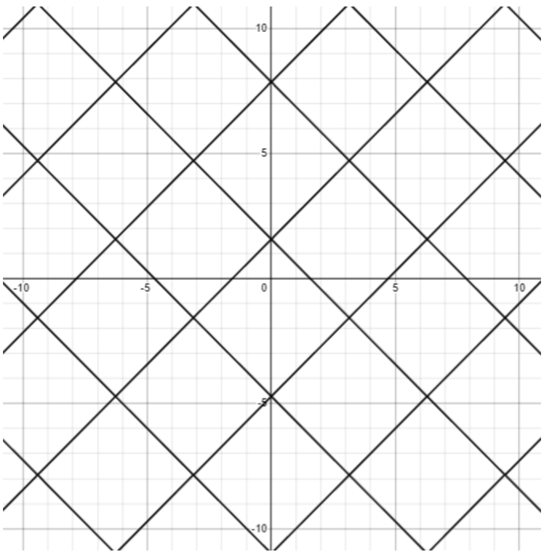


$$y = \tan(\sin(x))$$

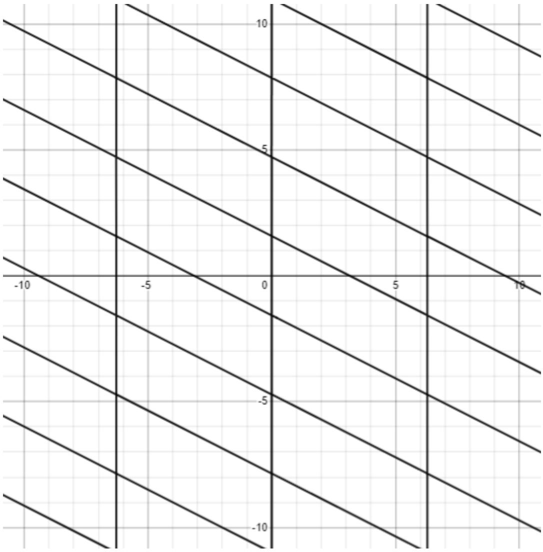


Demonstration - Implicit Trig Patterns. Observe the graph as you type each of the following inputs.

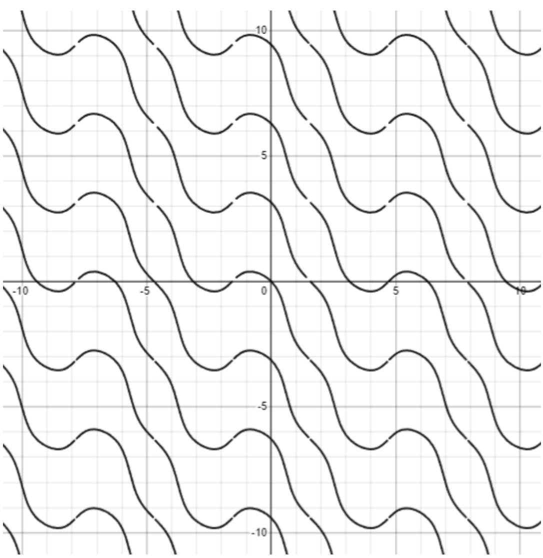
$$\sin(y) = \cos(x)$$



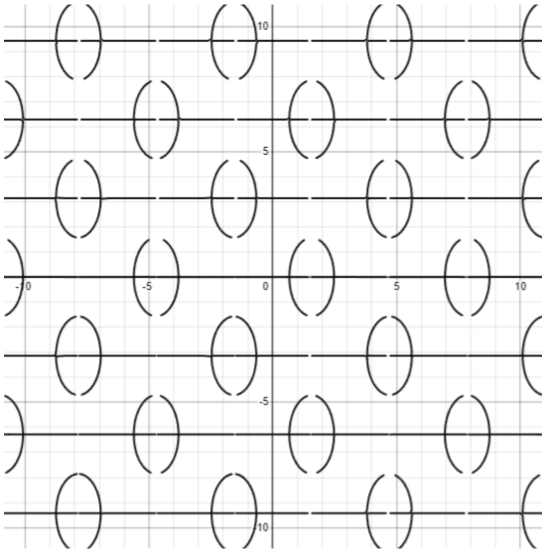
$$\sin(y) = \sin(x + y)$$



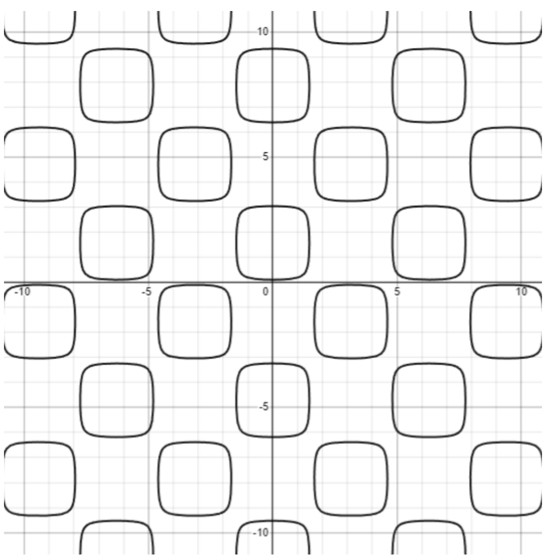
$$\sin(y)\tan(x) = \sin(x + y)$$



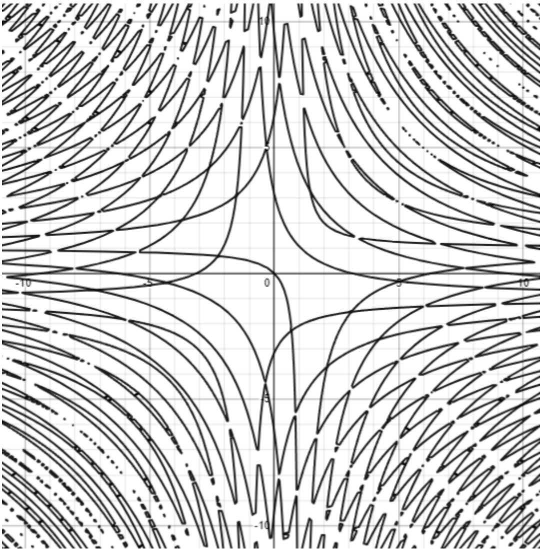
$$\sin(y)\tan(x) = \cos(x)\tan(y)$$



$$\sin(y)\cos(x) = 0.1$$



$$\sin(xy) = \sin(x+y)$$



Challenge. Create some interesting wallpapers using implicit trig patterns!