Justin Math | **Calculus**

First Edition by Justin Skycak

First edition.

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Limits and Derivatives

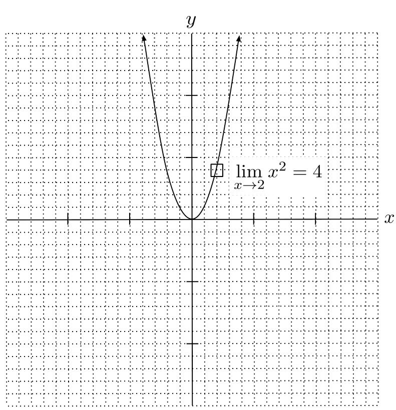
1.1 Evaluating Limits

The **limit** of a function $f(x)$, as x approaches some value a, is the value we would expect for $f(a)$ if we saw only the portion of the graph around (but not including) $x = a$. If the resulting value is L, then we denote the limit as follows:

$$\lim_{x \to a} f(x) = L$$

Limit vs Function Value

For example, for the function $f(x) = x^2$, the limit as $x \to 2$ is the same as the actual value of the function, which is 4.



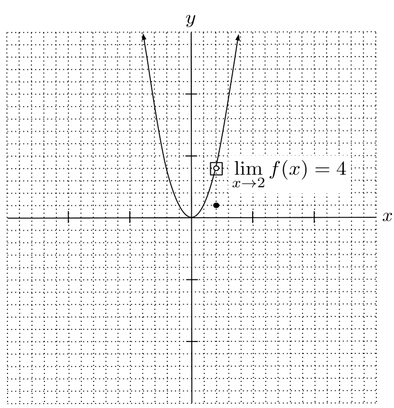
On the other hand, for the function

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

the limit as $x \to 2$ is not the same as the actual value of the function: the limit is 4, while the actual value of the function is $f(2) = 1$.

Remember, the limit is the value we would expect if we only saw the surrounding parts of the graph -- and in this graph, the surrounding y-values get closer and closer to 4 as the x-value gets closer and closer to 2.

Based on this, we expect that the y-value is 4, so we say the limit is 4, even though our expectation here is incorrect. The limit is still 4, and it is different from the actual function value $f(2) = 1$.



Continuity

Most of the functions we're familiar with from algebra are **continuous**, meaning that the actual output value $f(a)$ is the same as the limit as $x \to a$.

However, for **discontinuous** functions such as some piecewise functions and rational functions, the limit as $x \to a$ might be different from the actual output value $f(a)$.

If a function can be drawn in a single stroke, then it is continuous, and the limits are the same as the function values. However, if you need to pick up your pen at some point while drawing the function, then the function is discontinuous, and some limits might be different from the actual function values.

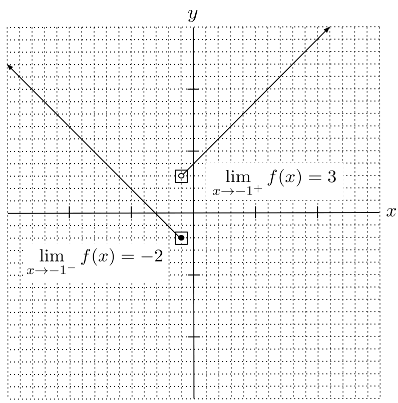
Existence of Limits

Sometimes, limits don't even exist. For example, for the function

$$f(x) = \begin{cases} -x-3 & \text{if } x \leq -1 \\ x+4 & \text{if } x > -1 \end{cases}$$

the limit comes out to different values, depending on whether we approach $x \to -1$ from the **left** (denoted $x \to -1^-$) or the **right** (denoted $x \to -1^+$).

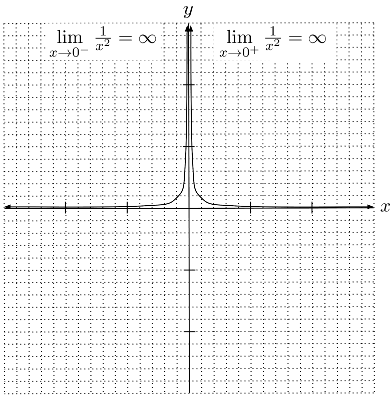
- Coming from the left, we are on the piece $f(x) = -x - 3$, so the limit is $-(-1) - 3 = -2$.
- Coming from the right, we are on the piece $f(x) = x + 4$, so the limit is $-1 + 4 = 3$.

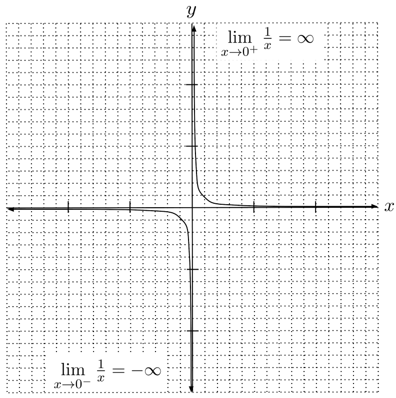


In general, a limit exists when its left and right limits are equal, and does not exist if its left and right limits are not equal.

For example, for the function $f(x) = \frac{1}{x^2}$, we have $\lim_{x \to 0} f(x) = \infty$ because $\lim_{x \to 0^-} f(x) = \infty$ and $\lim_{x \to 0^+} f(x) = \infty$.

On the other hand, for the function $f(x) = \frac{1}{x}$, we have that $\lim_{x \to 0} f(x)$ does not exist because $\lim_{x \to 0^-} f(x) = -\infty$ while $\lim_{x \to 0^+} f(x) = \infty$.

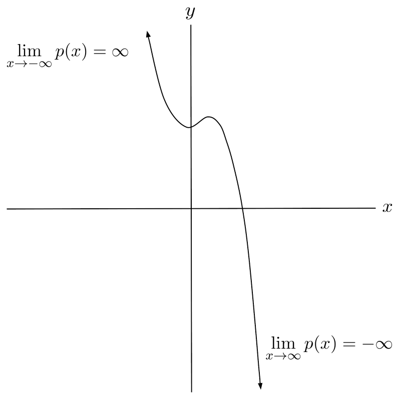




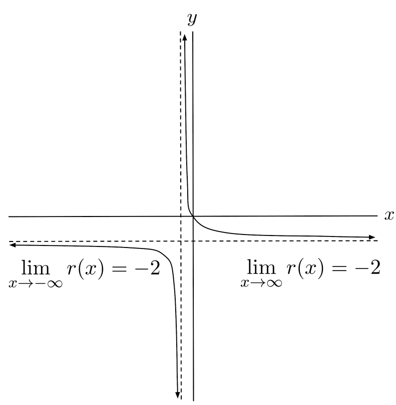
Limits at Infinity

One exception to the rule that left and right limits must be equal is **limits at infinity**, i.e. limits with $x \to \infty$ or $x \to -\infty$. In this case, it doesn't make sense to talk about a limit in more than one direction, because we can't choose numbers greater than ∞, and we can't choose numbers more negative than $-\infty$. As a consequence, limits with $x \to \infty$ are just taken as left limits ($x \to \infty^-$), and limits with $x \to -\infty$ are just taken as right limits ($x \to -\infty^+$).

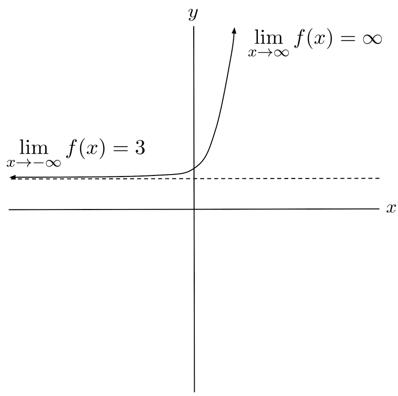
Limits at infinity can be thought of in terms of end behavior and horizontal asymptotes. For example, the polynomial $p(x) = -x^3 + x^2 + 5$ has end behavior $p(x) \to -\infty$ as $x \to \infty$, and $p(x) \to \infty$ as $x \to -\infty$. Its limits at infinity are then $\lim_{x\to\infty} p(x) = -\infty$, and $\lim_{x\to -\infty} p(x) = \infty$.



On the other hand, the end behavior of the rational function $r(x) = \frac{-2x}{x+1}$ consists of a horizontal asymptote $y = -2$. As a result, its limits at infinity are $\lim_{x\to\infty} r(x) = -2$, and $\lim_{x\to-\infty} r(x) = -2$.



The exponential function $f(x) = 2^x + 3$ has mixed end behavior: it blows up to infinity as $x \to \infty$, and settles down towards an asymptote $y = 3$ as $x \to -\infty$. Consequently, its limits at infinity are $\lim_{x \to \infty} f(x) = \infty$, and $\lim_{x \to -\infty} f(x) = 3$.



Indeterminate Form

Some limits like the one below can be difficult to think about graphically, because the function itself is difficult to graph.

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$$

At first sight, it's not clear how the function behaves as x approaches 0. We can't evaluate the function at $x = 0$ because it is undefined there, and it's not easy to see what happens as $x \to 0^+$ or $x \to 0^-$, since both the numerator and denominator go to 0. (Therefore, we say that the limit is **indeterminate** in its current form.)

Thankfully, algebraic tricks can often be used to simplify difficult limits into easier limits. In this case, if we multiply the numerator and denominator by the **conjugate** of the numerator, then we can simplify the limit to a point where we are able to evaluate the function at $x = 0$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \\
&= \lim_{x\to 0} \frac{(1+x)-1}{x\left(\sqrt{1+x}+1\right)} \\
&= \lim_{x\to 0} \frac{x}{x\left(\sqrt{1+x}+1\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} \\
&= \frac{1}{\sqrt{1+0}+1} \\
&= \frac{1}{2}
\end{aligned}$$

Similarly, to solve indeterminate limits where the numerator and denominator are both polynomials, we can often simplify the limit by factoring and canceling common factors:

$$\begin{aligned}
\lim_{x\to -2} \frac{x^2-x-6}{x+2} &= \lim_{x\to -2} \frac{(x+2)(x-3)}{x+2} \\
&= \lim_{x\to -2} x-3 \\
&= -2-3 \\
&= -5
\end{aligned}$$

Estimating Limits Numerically

Another trick for evaluating limits is thinking about them numerically. We can try substituting a number for x that is close to the intended limit in each direction, and doesn't make computations too hard.

For example, to evaluate the limit

$$\lim_{x \to -2} \frac{x^2 - x - 6}{x + 2}$$

numerically, we can approximate the left and right limits by substituting $x = -2.001$ and $x = -1.999$, respectively.

$$\lim_{x \to -2^-} \frac{x^2 - x - 6}{x + 2} \approx \frac{(-2.001)^2 - (-2.001) - 6}{-2.001 + 2} = -5.001$$

$$\lim_{x \to -2^+} \frac{x^2 - x - 6}{x + 2} \approx \frac{(-1.999)^2 - (-1.999) - 6}{-1.999 + 2} = -4.999$$

Both the left and right limits are approximately -5, so we would estimate the limit to be -5. Indeed, this matches the result we found earlier.

Likewise, to evaluate the limit

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$$

numerically, we can approximate the left and right limits numerically by substituting $x = -0.001$ and $x = 0.001$, respectively.

$$\lim_{x \to 0^-} \frac{\sqrt{1+x}-1}{x} \approx \frac{\sqrt{1+(-0.001)}-1}{-0.001} \approx 0.5001$$

$$\lim_{x \to 0^+} \frac{\sqrt{1+x}-1}{x} \approx \frac{\sqrt{1+0.001}-1}{0.001} \approx 0.4999$$

Both the left and right limits are approximately 0.5, so we would estimate the limit to be 0.5. Indeed, this matches the result we found earlier.

Caveat to Numerical Evaluation

One caveat to numerical evaluation is that it always results in decimal approximations, and if the actual limit value is irrational, it can be difficult to find the exact value of the limit.

In a simple case, we might be able to recognize that an approximation of 1.4142 actually corresponds to the value $\sqrt{2}$. However, in a trickier case, we might not be able to recognize that an approximation of 1.1547 actually corresponds to the value $\frac{2}{\sqrt{3}}$.

Exercises

Evaluate the indicated limits. If the limit does not exist, list the left and right limits separately (if applicable).

1) $f(x) = \frac{1}{3}x + 1$

$\lim_{x \to -6} f(x) =$ ___

2) $f(x) = 3x^2 + x - 4$

$\lim_{x \to 2} f(x) =$ ___

3) $f(x) = \begin{cases} x+9 & \text{if } x > -3 \\ x^2 + x & \text{if } x \leq -3 \end{cases}$

$\lim_{x \to -3} f(x) =$ ___

4) $f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 2 \\ 2x + 1 & \text{if } x < 2 \end{cases}$

$\lim_{x \to 2} f(x) =$ ___

5) $f(x) = \begin{cases} x^3 + x & \text{if } x > 5 \\ 2x - 3 & \text{if } x \leq 5 \end{cases}$

$\lim_{x \to 0} f(x) =$ ___

6) $f(x) = \begin{cases} 1 + \sin x & \text{if } x > \frac{\pi}{2} \\ \cos x & \text{if } x \leq \frac{\pi}{2} \end{cases}$

$\lim_{x \to \pi} f(x) =$ ___

7) $f(x) = \begin{cases} \sin x & \text{if } x \geq -\pi \\ \cos\left(x + \frac{\pi}{2}\right) & \text{if } x < -\pi \end{cases}$

$\lim_{x \to -\pi} f(x) =$ ___

8) $f(x) = \begin{cases} \tan x & \text{if } x \geq 1 \\ \ln x & \text{if } x < 1 \end{cases}$

$\lim_{x \to 1} f(x) =$ ___

9) $f(x) = \dfrac{1}{(x-3)^2}$

$\lim_{x \to 3} f(x) =$ ___

10) $f(x) = \dfrac{-3}{(x+2)^4}$

$\lim_{x \to -2} f(x) =$ ___

11) $f(x) = |\tan x|$

$\lim_{x \to \frac{\pi}{2}} f(x) =$ ___

12) $f(x) = \dfrac{1}{\sin x}$

$\lim_{x \to 0} f(x) =$ ___

13) $f(x) = \dfrac{1}{1 - \cos x}$

$\lim_{x \to 0} f(x) =$ ___

14) $f(x) = \begin{cases} -\frac{1}{x} & \text{if } x > 0 \\ \frac{1}{x} & \text{if } x \leq 0 \end{cases}$

$\lim_{x \to 0} f(x) =$ ___

15) $f(x) = \dfrac{x}{2x+1}$

$\lim_{x \to \infty} f(x) =$ ___

16) $f(x) = 4^{x-3}$

$\lim_{x \to -\infty} f(x) =$ ___

17) $f(x) = 3 + 2\sin x$

$\lim_{x \to \infty} f(x) =$ ___

18) $f(x) = \dfrac{2x}{\sqrt{1+x^2}}$

$\lim_{x \to -\infty} f(x) =$ ___

19) $f(x) = \log_2 x$

$\lim\limits_{x\to\infty} f(x) = \_\_\_$

20) $f(x) = \dfrac{\log_3 x}{\log_2 x}$

$\lim\limits_{x\to\infty} f(x) = \_\_\_$

21) $f(x) = \dfrac{x^2+3x+2}{x+1}$

$\lim\limits_{x\to -1} f(x) = \_\_\_$

22) $f(x) = \dfrac{2x^2-13x+15}{x-5}$

$\lim\limits_{x\to 5} f(x) = \_\_\_$

23) $f(x) = \dfrac{x^2-5x+6}{4-x^2}$

$\lim\limits_{x\to 2} f(x) = \_\_\_$

24) $f(x) = \dfrac{x^2-9}{(x-3)^3}$

$\lim\limits_{x\to 3} f(x) = \_\_\_$

25) $f(x) = \dfrac{x^3+x^2-x-1}{x^2-1}$

$\lim\limits_{x\to 1} f(x) = \_\_\_$

26) $f(x) = \dfrac{4-x^2}{x^2-4x+4}$

$\lim\limits_{x\to 2} f(x) = \_\_\_$

27) $f(x) = \dfrac{\sqrt{3+2x}-\sqrt{3}}{x}$

$\lim\limits_{x\to 0} f(x) = \_\_\_$

28) $f(x) = \dfrac{\sqrt{4+3x^2}-2}{x^2}$

$\lim\limits_{x\to 0} f(x) = \_\_\_$

29) $f(x) = \dfrac{\sqrt{x^2-1}-\sqrt{3}}{x+2}$

$\lim\limits_{x\to -2} f(x) = \_\_\_$

30) $f(x) = \dfrac{\sqrt{x+1}+1}{x^2}$

$\lim\limits_{x\to 0} f(x) = \_\_\_$

1.2 Limits by Logarithms, Squeeze Theorem, and Euler's Constant

A useful property of limits is that they can be brought inside continuous functions, i.e. the limit of a continuous function is the function of the limit.

For example, $\sqrt{x}$ is a continuous function, so to take the limit of the square root of some expression, we can first find the limit of the expression and then take the square root.

$$\lim_{x\to\infty}\sqrt{\frac{2x}{x+1}} = \sqrt{\lim_{x\to\infty}\frac{2x}{x+1}} = \sqrt{2}$$

We can do the same thing with other continuous functions, such as $\ln x$.

$$\begin{aligned}
&\lim_{x\to 2^+}\left[\ln(x^2-x-2)-\ln(x^2-3x+2)\right] \\
&= \lim_{x\to 2^+}\ln\left(\frac{x^2-x-2}{x^2-3x+2}\right) \\
&= \ln\left(\lim_{x\to 2^+}\frac{x^2-x-2}{x^2-3x+2}\right) \\
&= \ln\left(\lim_{x\to 2^+}\frac{(x-2)(x+1)}{(x-2)(x-1)}\right) \\
&= \ln\left(\lim_{x\to 2^+}\frac{x+1}{x-1}\right) \\
&= \ln 3
\end{aligned}$$

Exponential Limits

Logarithms in particular are useful for evaluating exponential limits, which have variables in both the limit and the base.

For example, to evaluate the limit

$$\lim_{x \to 0^+} x^{\ln x}$$

it is easiest to start by evaluating the logarithm of the limit.

$$\begin{aligned} &\ln\left(\lim_{x \to 0^+} x^{\ln x}\right) \\ &= \lim_{x \to 0^+} \ln\left(x^{\ln x}\right) \\ &= \lim_{x \to 0^+} (\ln x) \ln x \\ &= \lim_{x \to 0^+} (\ln x)^2 \\ &= \left(\lim_{x \to 0^+} \ln x\right)^2 \\ &= (-\infty)^2 \\ &= \infty \end{aligned}$$

Since we know the logarithm of the limit is ∞, the limit is just e raised to the power of ∞.

$$\lim_{x \to 0^+} x^{\ln x} = e^{\infty} = \infty$$

Using the same process, we can show that

$$\lim_{x \to 0^+} x^{(\ln x)^2} = e^{-\infty} = 0$$

because this time, the logarithm of the limit evaluates to $-\infty$.

$$\begin{aligned}
&\ln\left(\lim_{x \to 0^+} x^{(\ln x)^2}\right) \\
&= \lim_{x \to 0^+} \ln\left(x^{(\ln x)^2}\right) \\
&= \lim_{x \to 0^+} (\ln x)^2 \ln x \\
&= \lim_{x \to 0^+} (\ln x)^3 \\
&= \left(\lim_{x \to 0^+} \ln x\right)^3 \\
&= (-\infty)^3 \\
&= -\infty
\end{aligned}$$

Squeeze Theorem

Another useful trick for evaluating difficult limits is squeezing them between limits that are easier to evaluate.

For example, to evaluate the limit

$$\lim_{x \to \infty} \frac{\sin x}{x}$$

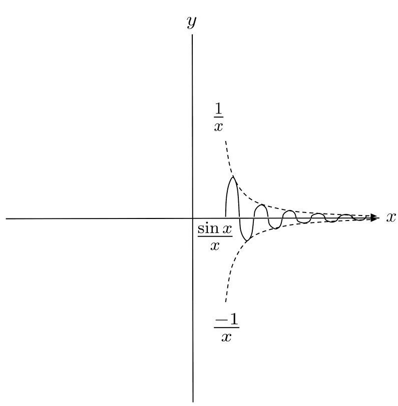
we can make use of the fact that $\sin x$ is bounded between -1 and 1. Then as $x \to \infty$ we have the following:

$$\begin{aligned} -1 \leq \quad & \sin x && \leq 1 \\ \frac{-1}{x} \leq \quad & \frac{\sin x}{x} && \leq \frac{1}{x} \\ \lim_{x\to\infty} \frac{-1}{x} \leq & \lim_{x\to\infty} \frac{\sin x}{x} && \leq \lim_{x\to\infty} \frac{1}{x} \\ 0 \leq & \lim_{x\to\infty} \frac{\sin x}{x} && \leq 0 \end{aligned}$$

The inequality states that the limit must be between 0 and 0, and the only number that is between 0 and 0 is 0 itself, so by the **squeeze theorem**, the limit must evaluate to 0.

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

In other words, the limit must be 0 because we squeezed it between two other limits, both of which evaluate to 0.



As another example, we can show that

$$\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$$

by performing a squeeze between the bounds of $\cos$:

$$\begin{gathered}\lim_{x\to 0^-} x(-1) \leq \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) \leq \lim_{x\to 0} x(1)\\ 0 \leq \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) \leq 0\end{gathered}$$

Euler's Constant

Lastly, **Euler's constant** e can be expressed as the following limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

It also holds as $n \to -\infty$:

$$e = \lim_{n\to-\infty}\left(1+\frac{1}{n}\right)^n$$

Substituting $x = \frac{1}{n}$, we can also express the limit as

$$e = \lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}.$$

Knowing the above limit forms of Euler's constant allows us to compute limits that are in a similar form. For example, to compute the limit

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{3n}$$

we can make a substitution that results in $\frac{2}{n}=\frac{1}{u}$. Then $n=2u$, and $n\to\infty$ translates to $u\to\infty$, and the limit becomes computable in terms of Euler's constant:

$$\begin{aligned}
&\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{3n}\\
&=\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{3(2u)}\\
&=\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{6u}\\
&=\lim_{u\to\infty}\left[\left(1+\frac{1}{u}\right)^{u}\right]^{6}\\
&=\left[\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{u}\right]^{6}\\
&=e^{6}
\end{aligned}$$

Similarly, to compute the limit

$$\lim_{x\to 0}\left(1-3x\right)^{\frac{5}{x}}$$

we can make a substitution that results in $-3x = u$. Then $x = -\frac{1}{3}u$, and $x \to 0$ translates to $u \to 0$, and the limit becomes computable in terms of Euler's constant:

$$\begin{aligned}
&\lim_{x\to 0}(1-3x)^{\frac{5}{x}} \\
&= \lim_{u\to 0}(1+u)^{\frac{5}{-\frac{1}{3}u}} \\
&= \lim_{u\to 0}(1+u)^{-\frac{15}{u}} \\
&= \lim_{u\to 0}\left[(1+u)^{\frac{1}{u}}\right]^{-15} \\
&= \left[\lim_{u\to 0}(1+u)^{\frac{1}{u}}\right]^{-15} \\
&= e^{-15}
\end{aligned}$$

Exercises

Evaluate the following limits using logarithms.

1) $\lim\limits_{x\to 0^+} x^{(\ln x)^3}$

2) $\lim\limits_{x\to 0^+} x^{(\ln x)^4}$

3) $\lim\limits_{x\to 0^+} x^{\frac{1}{\ln x}}$

4) $\lim\limits_{x\to 0^+} x^{\frac{1}{(\ln x)^2}}$

Evaluate the following limits using the squeeze theorem.

5) $\displaystyle\lim_{x\to\infty} \frac{\sin x + \cos x}{\ln x}$

6) $\displaystyle\lim_{x\to 0^+} \sqrt{x}\cos\left(\frac{1}{1-e^x}\right)$

7) $\displaystyle\lim_{x\to\infty} \frac{3x + \sin x}{\sqrt{3x^2 - 1}}$

8) $\displaystyle\lim_{x\to 1^+} (\ln x)\sin\left(\frac{1}{\ln x}\right)$

9) $\displaystyle\lim_{x\to -\infty} \frac{4x + 3\cos x}{2x+1}$

10) $\displaystyle\lim_{x\to\infty} \frac{2x + x\sin x}{\sqrt{x^3 - 4}}$

Evaluate the following limits using Euler's constant.

11) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{4n}$

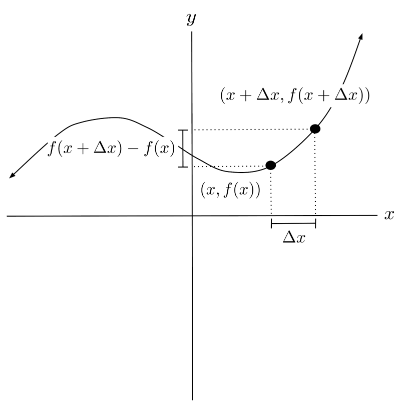
12) $\displaystyle\lim_{x\to 0} \left(1 + \frac{2}{3}x\right)^{\frac{1}{2x}}$

13) $\displaystyle\lim_{n\to -\infty} \left(1 + \frac{5}{3n}\right)^{-2n}$

14) $\displaystyle\lim_{x\to 0} \left(1 + \frac{x}{\pi}\right)^{\frac{e}{x}}$

1.3 Derivatives and the Difference Quotient

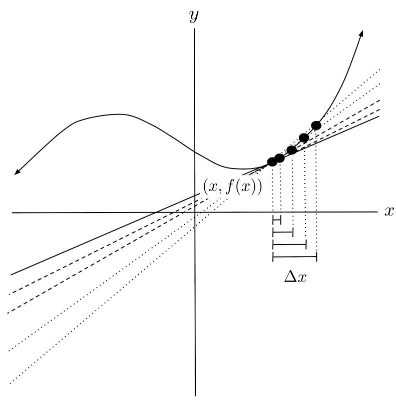
The **derivative** of a function is the function's slope at a particular point. We can approximate the derivative at a point $(x, f(x))$ by choosing another nearby point on the function, and computing the slope. If we increase the input x by a small amount Δx, then we reach an x-coordinate of $x + \Delta x$, and the corresponding point on the function is $(x + \Delta x, f(x + \Delta x))$.



We compute the slope between the points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$, and simplify.

$$
\begin{aligned}
m &= \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x} \\
&= \frac{f(x + \Delta x) - f(x)}{\Delta x}
\end{aligned}
$$

The above expression is called the **difference quotient** of the function f. As the point $(x + \Delta x, f(x + \Delta x))$ gets closer and closer to the point $(x, f(x))$, the difference quotient becomes a better and better approximation for the exact slope at $(x, f(x))$.



Thus, we can compute derivative, which is the exact slope at $(x, f(x))$, by taking the limit as the second point approaches the first point. In other words, the derivative is the limit of the difference quotient as the difference Δx between the two input x-values approaches 0.

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The derivative of the function f at the point $(x, f(x))$ is indicated by the notation $\frac{d}{dx} f(x)$. However, to simplify notation, we often write the derivative as $f'(x)$ instead of $\frac{d}{dx} f(x)$.

Demonstration

As an example, we'll use the difference quotient show that the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + (2\Delta x)x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(2\Delta x)x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2x + \Delta x \\ &= 2x + 0 \\ &= 2x \end{aligned}$$

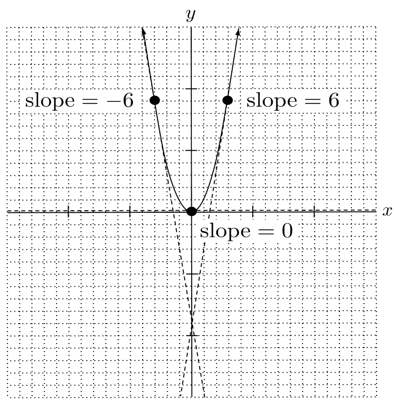
This means that the slope of $f(x) = x^2$ at any point $(x, f(x))$ is given by $f'(x) = 2x$.

In particular, the slope at $x = -3$ is given by $f'(-3) = -6$, the slope at $x = 0$ is given by $f'(0) = 0$, and the slope at $x = 3$ is given by $f'(3) = 6$.

Looking at the graph, these values make sense:

- At $x = -3$, the graph is falling down at a steep angle, which matches the negative derivative $f'(-3) = -6$.

- At $x = 0$, the graph is flat at the bottom of a valley, which matches the derivative $f'(0) = 0$.
- At $x = 3$, the graph is climbing up at a steep angle, which matches the positive derivative $f'(3) = 6$.



The values for the derivative also make sense numerically:

- If we start at the point $(-3, 9)$ and pick another point $(-2.999, 8.994001)$ on the function $f(x) = x^2$, the slope between the two points is $\frac{8.994001-9}{-2.999-(-3)} = -5.999$, which approximates our derivative value $f'(-3) = -6$.
- If we start at the point $(0, 0)$ and pick another point $(0.001, 0.000001)$ on the function $f(x) = x^2$, the slope between the two points is $\frac{0.000001-0}{0.001-0} = 0.001$, which

approximates our derivative value $f'(0) = 0$.

- If we start at the point $(3, 9)$ and pick another point $(3.001, 9.006001)$ on the function $f(x) = x^2$, the slope between the two points is $\frac{9.006001-9}{3.001-3} = 6.001$, which approximates our derivative value $f'(3) = 6$.

Exercises

Use the difference quotient to differentiate the following functions.

1) $f(x) = 5x$

2) $f(x) = -3x + 1$

3) $f(x) = x^2$

4) $f(x) = 7x^2$

5) $f(x) = x^2 - x$

6) $f(x) = x^3$

7) $f(x) = 2\sqrt{x}$

8) $f(x) = \sqrt{3x}$

9) $f(x) = \frac{1}{x}$

10) $f(x) = \frac{1}{2x+1}$

1.4 Power Rule

It can be a pain to evaluate the difference quotient every time we want to take the derivative of a function. Luckily, there are some patterns in derivatives that allow us to compute derivatives without having to go through all the steps of computing the limit of the difference quotient.

One such pattern is the **power rule**, which tells us that the derivative of a function $f(x) = x^n$, where n is some constant number, is given by $f'(x) = nx^{n-1}$. Several examples are shown below.

$$(x^5)' = 5x^4 \qquad (x)' = (x^1)' = 1x^0 = 1 \qquad (x^{-3})' = -3x^{-4}$$

Further Applications

We can also use the power rule to differentiate constants and radical expressions.

$$(1)' = (x^0)' = 0x^{-1} = 0$$

$$(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

$$\left(\sqrt[3]{x^5}\right)' = \left(x^{\frac{5}{3}}\right)' = \tfrac{5}{3}x^{\frac{2}{3}} = \tfrac{5}{3}\sqrt[3]{x^2}$$

When a term is multiplied by some constant number, we can move the number outside of the derivative, i.e. we can take the derivative of the term and multiply it by that number.

$$(5)' = (5x^0)' = 5(x^0)' = 5(0) = 0$$

$$(-3\sqrt{x})' = -3\left(\sqrt{x}\right)' = -3\left(\tfrac{1}{2\sqrt{x}}\right) = -\tfrac{3}{2\sqrt{x}}$$

$$\left(\tfrac{1}{4}\sqrt[3]{x^5}\right)' = \tfrac{1}{4}\left(\sqrt[3]{x^5}\right)' = \tfrac{1}{4}\left(\tfrac{5}{3}\sqrt[3]{x^2}\right) = \tfrac{5}{12}\sqrt[3]{x^2}$$

In general, for any number c, we have

$$(cx^n)' = c(x^n)' = cnx^{n-1}.$$

When we have a sum or difference of terms, we can apply the power rule to each term individually.

$$\begin{aligned}
&\left(\frac{2}{3}x^3 - 5\sqrt{x} + \frac{1}{4x^2} - 7\right)' \\
&= \left(\frac{2}{3}x^3\right)' - (5\sqrt{x})' + \left(\frac{1}{4x^2}\right)' - (7)' \\
&= \frac{2}{3}\left(x^3\right)' - 5\left(\sqrt{x}\right)' + \frac{1}{4}\left(\frac{1}{x^2}\right)' - 7\,(1)' \\
&= \frac{2}{3}\left(3x^2\right) - 5\left(\frac{1}{2\sqrt{x}}\right) + \frac{1}{4}\left(-\frac{2}{x^3}\right) - 7\,(0) \\
&= 2x^2 - \frac{5}{2\sqrt{x}} - \frac{1}{2x^3}
\end{aligned}$$

Derivation

To see why the power rule works, we can compute the derivative for x^n using the difference quotient.

$$\begin{aligned}
(x^n)' &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)(x+\Delta x)\cdots(x+\Delta x) - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + \left[\text{other terms of at least } (\Delta x)^2\right] - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \left[\text{other terms of at least } (\Delta x)^2\right]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + (\Delta x)^2(\text{other terms})}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[nx^{n-1} + (\Delta x)(\text{other terms})\right] \\
&= nx^{n-1} + (0)(\text{other terms}) \\
&= nx^{n-1} + 0 \\
&= nx^{n-1}
\end{aligned}$$

Exercises

Use the power rule to differentiate the following functions.

1) $f(x) = x^{\frac{4}{3}}$

2) $f(x) = \frac{1}{x^6}$

3) $f(x) = 4\sqrt{x^3}$

4) $f(x) = -\frac{1}{2}x^{\frac{4}{5}}$

5) $f(x) = \frac{1}{\sqrt{x}} + 3$

6) $f(x) = x^{\frac{1}{72}} - x^2$

7) $f(x) = \frac{3}{\sqrt{x^5}} - 3\sqrt{x}$

8) $f(x) = 2x^{3.1} + \frac{1}{2}x^{102}$

9) $f(x) = x^{\sqrt{2}} + \sqrt{3}x^{\sqrt{3}} - \frac{\sqrt{2}}{x^{\sqrt{2}}}$

10) $f(x) = \frac{e}{x^{\pi}} + \pi x^{e} - \pi x^{\frac{e}{\pi}}$

1.5 Chain Rule

The **chain rule** tells us how to take derivatives of compositions of functions. Informally, it says that we can "forget" about the inside of a function when we take the derivative, as long as we multiply by the derivative of the inside afterwards.

For example, to differentiate $(x^2+1)^{100}$, we can use the power rule, as long as we multiply by the derivative of the inside (x^2+1) afterwards.

$$\begin{aligned}\left[(x^2+1)^{100}\right]' &= 100(x^2+1)^{99}(x^2+1)' \\ &= 100(x^2+1)^{99}(2x) \\ &= 200x(x^2+1)^{99}\end{aligned}$$

Substitution

More precisely, the chain rule states that we can make a substitution u for an expression of x, as long as we multiply by the derivative of the substitution afterwards.

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

To differentiate the function $f(x)=(x^2+1)^{100}$, we substituted $u=x^2+1$ to simplify the function to $f(u)=u^{100}$.

$$\begin{aligned} \frac{df}{du} &= 100u^{99} \\ &= 100(x^2+1)^{99} \end{aligned} \qquad \frac{du}{dx} = 2x \qquad \begin{aligned} \frac{df}{dx} &= \frac{df}{du}\frac{du}{dx} \\ &= 100(x^2+1)^{99}(2x) \\ &= 200x(x^2+1)^{99} \end{aligned}$$

Intuitively, the chain rule says that we can cancel derivatives just like we cancel fractions.

$$\frac{df}{dx} = \frac{df}{\cancel{du}}\frac{\cancel{du}}{dx}$$

We can extend this to an unlimited number of substitutions, building a "chain" of cancellations.

$$\frac{df}{dx} = \frac{df}{\cancel{du_1}}\frac{\cancel{du_1}}{\cancel{du_2}}\frac{\cancel{du_2}}{\cancel{du_3}}\cdots\frac{\cancel{du_{n-1}}}{\cancel{du_n}}\frac{\cancel{du_n}}{dx}$$

For example, to differentiate the function $f(x) = (((x^3+1)^4+1)^5+1)^6$ we can proceed one layer at a time.

$$\begin{aligned} &\left[((((x^3+1)^4+1)^5+1)^6\right]' \\ &= 6(((x^3+1)^4+1)^5+1)^5 \\ &\quad \cdot \left[((x^3+1)^4+1)^5+1\right]' \end{aligned}$$

$$
\begin{aligned}
&= 6(((x^3+1)^4+1)^5+1)^5 \\
&\quad \cdot 5((x^3+1)^4+1)^4 \\
&\quad \cdot \left[(x^3+1)^4+1\right]'
\end{aligned}
$$

$$
\begin{aligned}
&= 6(((x^3+1)^4+1)^5+1)^5 \\
&\quad \cdot 5((x^3+1)^4+1)^4 \\
&\quad \cdot 4(x^3+1)^3 \\
&\quad \cdot \left[x^3+1\right]'
\end{aligned}
$$

$$
\begin{aligned}
&= 6(((x^3+1)^4+1)^5+1)^5 \\
&\quad \cdot 5((x^3+1)^4+1)^4 \\
&\quad \cdot 4(x^3+1)^3 \\
&\quad \cdot 3x^2
\end{aligned}
$$

Exercises

Use the chain rule to find the derivatives of the following functions.

1) $f(x) = (2x^2+1)^3$

2) $f(x) = (x^4 - x^2)^8$

3) $f(x) = \sqrt{x^2+1}$

4) $f(x) = \sqrt{(2x+1)^2+3}$

5) $f(x) = \frac{1}{3x-2}$

6) $f(x) = \frac{7}{(x^2-3)^2}$

7) $f(x) = \frac{2}{\sqrt{1-x^5}}$

8) $f(x) = \sqrt{\sqrt{x}+1}$

9) $f(x) = \left(\sqrt{x}+1\right)^5$

10) $f(x) = \left(x^{\frac{3}{2}} + x^{\frac{4}{3}}\right)^{\frac{5}{4}}$

1.6 Properties of Derivatives

We know that when differentiating polynomials, we can differentiate each term individually. But why are we able to do this? Does multiplication work the same way? What about division? We answer these questions in this chapter.

Sum Rule

First of all, we are able to differentiate each term in a polynomial individually, because in general, derivatives can be separated over addition. The derivative of a sum, is the sum of derivatives of individual terms.

To see why this is true, we can look at what happens in the difference quotient when we take the derivative of the sum of two functions. We are able to rearrange the difference quotient into the sum of difference quotients of the two functions, which shows that the derivative of the sum is just the sum of the derivatives.

$$\begin{aligned}
(f(x)+g(x))' &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)+g(x+\Delta)-(f(x)+g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)+g(x+\Delta)-g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta)-g(x)}{\Delta x} \\
&= f'(x)+g'(x)
\end{aligned}$$

Constant Multiple Rule

Another useful property of derivatives is that constants can be moved outside the derivative.

$$(2x^3)' = 2(x^3)' = 2(3x^2) = 6x^2$$

Combining this with the power rule, we can differentiate entire polynomial expressions.

$$\begin{aligned}(3x^4 + x^2 - 2x + 1)' &= (3x^4)' + (x^2)' + (-2x)' + (1)' \\ &= 3(x^4)' + (x^2)' - 2(x)' + (1)' \\ &= 3(4x^3) + (2x) - 2(1) + 0 \\ &= 12x^3 + 2x - 2\end{aligned}$$

To see why we can move constants outside the derivative, we can inspect what happens in the difference quotient when we take the derivative of a function multiplied by a constant. The constant factors out, and we can write the result as the product of the constant and the derivative.

$$\begin{aligned}(cf(x))' &= \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} c\frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= c\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= cf'(x)\end{aligned}$$

Product Rule

Taking the derivative of a product, perhaps surprisingly, results in a sum. For each term that is multiplied in the product, a copy of the product is added in the sum, with the particular term replaced by its derivative.

$$\left[(3x^2+2)(x+3)^2(2x+1)^5\right]'$$

$$\begin{aligned} &= (3x^2+2)'(x+3)^2(2x+1)^5 \\ &\quad + (3x^2+2)\left[(x+3)^2\right]'(2x+1)^5 \\ &\quad + (3x^2+2)(x+3)^2\left[(2x+1)^5\right]' \end{aligned}$$

$$\begin{aligned} &= (6x)(x+3)^2(2x+1)^5 \\ &\quad + (3x^2+2)\left[2(x+3)\right](2x+1)^5 \\ &\quad + (3x^2+2)(x+3)^2\left[10(2x+1)^4\right] \end{aligned}$$

$$= (x+3)(2x+1)^4 \begin{bmatrix} 6x(x+3)(2x+1) \\ +(3x^2+2)(2)(2x+1) \\ +(3x^2+2)(x+3)(10) \end{bmatrix}$$

$$= (x+3)(2x+1)^4(54x^3+138x^2+46x+64)$$

To see why this works, we can look at what happens in the difference quotient when we take the derivative of the product of two functions. We are able to rearrange the difference quotient into the sum of the difference quotients of the two functions, with each difference quotient multiplied by the other function. This shows that the derivative of the product is a sum of copies of the product, each with one particular term replaced by its derivative.

$$\begin{aligned}
(f(x)g(x))' &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\left[g(x+\Delta x) - g(x)\right] + g(x)\left[f(x+\Delta x) - f(x)\right]}{\Delta x} \\
&= \lim_{\Delta x \to 0} f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= f(x+0) \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x) \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= f(x)g'(x) + g(x)f'(x) \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

Quotient Rule

To take the derivative of a quotient, we can use the product rule in conjunction with the power rule and chain rule.

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x)g(x)^{-1}\right)' \\
&= f'(x)g(x)^{-1} + f(x)\left(g(x)^{-1}\right)' \\
&= f'(x)g(x)^{-1} + f(x)\left(-g(x)^{-2}g'(x)\right) \\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}$$

Applying this formula can save us the work of combining fractions after differentiating.

$$\begin{aligned}\left(\frac{x^2}{2x+1}\right)' &= \frac{(x^2)'(2x+1)-(x^2)(2x+1)'}{(2x+1)^2}\\ &= \frac{(2x)(2x+1)-(x^2)(2)}{(2x+1)^2}\\ &= \frac{2x^2+2x}{(2x+1)^2}\end{aligned}$$

Exercises

Use the properties of derivatives to differentiate the following functions.

1) $f(x)=\sqrt{x}(x+2)^2$

2) $f(x)=(x-3)^2(x+1)^3$

3) $f(x)=x^2(x+1)^3$

4) $f(x)=(2x+3)^4(x^2-5)^3\sqrt{x}$

5) $f(x)=\dfrac{x+1}{x+2}$

6) $f(x)=\dfrac{x}{x^2+1}$

7) $f(x)=\dfrac{x^2+2x+3}{2x^2-5}$

8) $f(x)=\dfrac{x^3-2x}{x^4-1}$

9) $f(x)=\dfrac{x^3(x+1)^4}{(x-1)^3}$

10) $f(x)=\dfrac{8x^2-\sqrt{x}}{\sqrt{x}+2}$

1.7 Derivatives of Non-Polynomial Functions

In this chapter, we introduce rules for the derivatives of exponential, logarithmic, trigonometric, and inverse trigonometric functions. Although it's possible to compute each derivative using the difference quotient, it will take a long time to compute derivatives during calculus problems if we have to start from scratch with the difference quotient process every time -- so it's advantageous to remember the derivative rules. The derivative rules are to calculus, what the multiplication table is to arithmetic.

Natural Logarithm

We start with the **natural logarithm**, which has the derivative $(\ln x)' = \frac{1}{x}$. To see where this formula comes from, we can start by writing and simplifying the difference quotient for $\ln x$.

$$\begin{aligned}
(\ln x)' &= \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\ln(x+\Delta x) - \ln x\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(\frac{x+\Delta x}{x}\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right) \\
&= \lim_{\Delta x \to 0} \ln\left[\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}\right] \\
&= \ln\left[\lim_{\Delta x \to 0}\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}\right]
\end{aligned}$$

Does the limit inside the natural log look familiar? Remember that the constant e can be written as the following limit:

$$e = \lim_{u \to 0}(1+u)^{\frac{1}{u}}$$

If we substitute $u = \frac{\Delta x}{x}$ and simplify/rearrange, then we can come up with an expression for the limit inside the natural log. (The limit as $\frac{\Delta x}{x} \to 0$ can be thought of as $\Delta x \to 0x$, which is the same as $\Delta x \to 0$.)

$$e = \lim_{\frac{\Delta x}{x}\to 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\frac{\Delta x}{x}}}$$

$$e = \lim_{\Delta x\to 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\frac{\Delta x}{x}}}$$

$$e = \lim_{\Delta x\to 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

$$e = \left[\lim_{\Delta x\to 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}\right]^{x}$$

$$e^{\frac{1}{x}} = \lim_{\Delta x\to 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}$$

Substituting this expression into the natural log, we find that $(\ln x)' = \frac{1}{x}$.

$$\begin{aligned}(\ln x)' &= \ln\left[\lim_{\Delta x\to 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}\right]\\ &= \ln\left(e^{\frac{1}{x}}\right)\\ &= \frac{1}{x}\end{aligned}$$

Knowing this, we can use the chain rule to find the derivative of any natural log function.

$$\left(\ln(2x^2+x)\right)' = \frac{1}{2x^2+x}(2x^2+x)' = \frac{4x+1}{2x^2+x}$$

$$\left((\ln x)^2\right)' = 2(\ln x)(\ln x)' = 2(\ln x)\left(\frac{1}{x}\right) = \frac{2\ln x}{x}$$

General Logarithms

To differentiate a logarithmic function other than the natural logarithm, we can use the change-of-base formula to rewrite the logarithmic function in terms of natural logarithms.

For example, to find the derivative of $\log_2 x$, we can convert it into $\frac{\ln x}{\ln 2}$ and then take the derivative.

$$(\log_2 x)' = \left(\frac{\ln x}{\ln 2}\right)' = \frac{1}{\ln 2}(\ln x)' = \frac{1}{\ln 2}\left(\frac{1}{x}\right) = \frac{1}{x\ln 2}$$

In general, performing this procedure on any function of the form $\log_a x$ where a is a constant, we find that $(\log_a x)' = \frac{1}{x\ln a}$.

Exponential Functions

Next, we cover **exponential functions**. The exponential function e^x is very elegant in calculus, because its derivative is simply itself, $(e^x)' = e^x$.

To see why this is, we can start with the equation $f(x) = e^x$, then take the logarithm and derivative of both sides, and finally solve for $f'(x)$.

$$\begin{aligned} f(x) &= e^x \\ \ln f(x) &= x \\ (\ln f(x))' &= (x)' \\ \frac{1}{f(x)} f'(x) &= 1 \\ f'(x) &= f(x) \\ f'(x) &= e^x \end{aligned}$$

Now that we know the derivative of e^x, we can use the chain rule to find the derivative of any exponential function.

$$\left(e^{x^2}\right)' = e^{x^2}(x^2)' = e^{x^2}(2x)$$

$$\left(\sqrt{e^x}\right)' = \left((e^x)^{\frac{1}{2}}\right)' = \left(e^{\frac{1}{2}x}\right)' = \frac{1}{2}e^{\frac{1}{2}x}$$

If we want to take the derivative of an exponential function whose base is not e, we can rewrite the exponential function so that its base is e, and then differentiate using the chain rule. For example, since $2 = e^{\ln 2}$, we see that

$$2^x = \left(e^{\ln 2}\right)^x = e^{(\ln 2)x}.$$

Now that we have a function which has base e, we can use the chain rule to find the derivative.

$$(2^x)' = \left(e^{(\ln 2)x}\right)' = e^{(\ln 2)x} \ln 2$$

Using the fact that $2^x = e^{(\ln 2)x}$, we can simplify the result a bit to look like the original function.

$$(2^x)' = e^{(\ln 2)x} \ln 2 = 2^x \ln 2$$

In general, performing this procedure on any function of the form a^x where a is a constant, we find that $(a^x)' = a^x \ln a$.

Trigonometric Functions

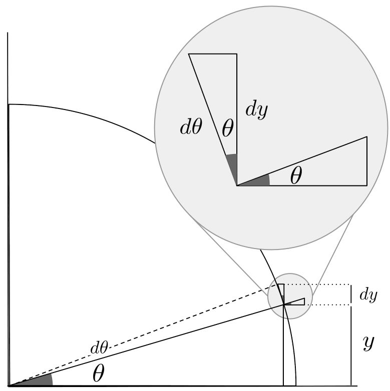
Now, let's talk about **trig functions**. Their derivatives are shown below.

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x \qquad (\csc x)' = -\csc x \cot x$$

To see why the derivative of sine is cosine, consider a section of the unit circle, where $y = \sin\theta$. If we increase θ by an infinitesimally small amount $d\theta$, the additional arc length $d\theta$ matches the hypotenuse of a triangle that has a leg dy adjacent to an angle θ. In this triangle, we have $\cos\theta = \frac{dy}{d\theta} = (\sin\theta)'$.



Furthermore, we can use the derivative of sine in conjunction with the identities $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ to compute the derivative of cosine.

$$\begin{aligned}
\cos x &= \sin\left(\frac{\pi}{2} - x\right) \\
(\cos x)' &= \left[\sin\left(\frac{\pi}{2} - x\right)\right]' \\
&= \cos\left(\frac{\pi}{2} - x\right)\left(\frac{\pi}{2} - x\right)' \\
&= \cos\left(\frac{\pi}{2} - x\right)(-1) \\
&= (\sin x)(-1) \\
&= -\sin x
\end{aligned}$$

The fundamental trig derivatives are $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$; all the other trig derivatives come from using them.

For example, to see that $(\sec x)' = \sec x \tan x$, we express $\sec x$ as $\frac{1}{\cos x}$, take the derivative using the chain rule, and simplify.

$$\begin{aligned}
(\sec x)' &= \left(\frac{1}{\cos x}\right)' \\
&= -\frac{1}{\cos^2 x}(\cos x)' \\
&= -\frac{1}{\cos^2 x}(\cos x)' \\
&= -\frac{1}{\cos^2 x}(-\sin x) \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}$$

Mnemonics

However, it will take a long time to compute derivatives if we have to start from scratch with the above process every time, so it's advantageous to remember the table of trig derivatives.

To make it easier to remember the table, think about three key trends in the table: functions have buddies, "co" functions turn negative, and derivatives of functions other than $\sin$ and $\cos$ have two terms.

More precisely, the functions $\sin$ and $\cos$ are buddies because the derivative of $\sin$ contains $\cos$ and the derivative of $\cos$ contains $\sin$. Likewise, $\sec$ and $\tan$ are buddies because the derivative of $\sec$ contains $\tan$ and the derivative of $\tan$ contains $\sec$, and $\csc$ and $\cot$ are buddies because the derivative of $\csc$ contains $\cot$ and the derivative of $\cot$ contains $\csc$.

“Co” functions include $\cos$, $\csc$, and $\cot$, and each of their derivatives has a negative sign, whereas the other functions do not have a negative sign in their derivatives.

Lastly, if we think of squared terms as two terms being multiplied together, then $\sin$ and $\cos$ are the only functions whose derivatives consist of a single term. For example, the derivative of $\sec$ is the product of two terms $\sec$ and $\tan$, and the derivative of $\tan$ is $\sec^2$ which can be interpreted as the product of two terms $\sec$ and $\sec$. On the other hand, the derivative of $\sin$ is just a single term, $\cos$.

Just as we did for exponential and logarithmic derivatives, we can use the chain rule to take the derivative of any trig function.

$$(\sin(\ln x))' = \cos(\ln x)(\ln x)' = \frac{\cos(\ln x)}{x}$$

$$\left(\sec^2 x\right)' = (2\sec x)(\sec x)' = 2\sec^2 x \tan x$$

Inverse Trigonometric Functions

Now that we know the derivatives of trig functions, we can use them to find the derivatives of inverse trig functions, which are shown below.

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

To see where these derivatives come from, we can proceed in the same way as earlier when we used the logarithmic function to find the derivative of the exponential function. We start with the equation $f(x) = \arcsin x$, then take the $\sin$ and derivative of both sides, and finally solve for $f'(x)$.

$$\begin{aligned}
f(x) &= \arcsin x \\
\sin f(x) &= x \\
(\sin f(x))' &= (x)' \\
(\cos f(x))f'(x) &= 1 \\
f'(x) &= \frac{1}{\cos f(x)} \\
f'(x) &= \frac{1}{\cos(\arcsin x)}
\end{aligned}$$

To simplify the denominator, we solve for $\cos\theta$ in the identity $\sin^2\theta + \cos^2\theta = 1$ with $\theta = \arcsin x$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \cos^2\theta &= 1 - \sin^2\theta \\ \cos\theta &= \pm\sqrt{1 - \sin^2\theta} \end{aligned}$$

We only need to consider the positive root because $\cos$ is always nonnegative on the range of $\arcsin$, which is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Substituting $\theta = \arcsin x$, our expression simplifies.

$$\begin{aligned} \cos\left(\arcsin x\right) &= \sqrt{1 - \sin^2\left(\arcsin x\right)} \\ \cos\left(\arcsin x\right) &= \sqrt{1 - x^2} \end{aligned}$$

Substituting the above identity in the denominator of our derivative expression, we obtain the final result.

$$\begin{aligned} f'(x) &= \frac{1}{\cos(\arcsin x)} \\ f'(x) &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

The rest of the inverse trig derivatives can be computed by the same process. Now, we can use the chain rule to take the derivative of any inverse trig function.

$$(\arctan(e^x))' = \frac{1}{1+(e^x)^2}(e^x)' = \frac{e^x}{1+e^{2x}}$$

$$\left((\arcsin x)^2\right)' = 2(\arcsin x)(\arcsin x)' = \frac{2\arcsin x}{\sqrt{1-x^2}}$$

Exercises

Compute the derivative of each function.

1) $f(x) = \dfrac{1}{\ln x}$

2) $f(x) = e^{1+\tan x}$

3) $f(x) = 2^{\sin x}$

4) $f(x) = \arctan\left(\dfrac{1}{x}\right)$

5) $f(x) = \ln\left(\cos x^2\right)$

6) $f(x) = \log_3(\sin 2x)$

7) $f(x) = \arccos(\log_5 x)$

8) $f(x) = \sin^2(x)\cos^3(x)$

9) $f(x) = \dfrac{1+e^x}{1-e^x}$

10) $f(x) = \dfrac{\arcsin x}{\arccos x}$

11) $f(x) = \arctan\left(\dfrac{x}{1+e^x}\right)$

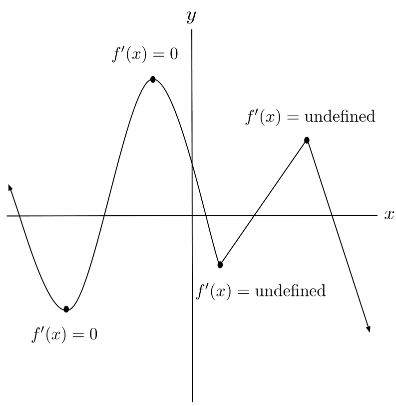
12) $f(x) = \arccos(x)\arcsin^2(x)$

13) $f(x) = \dfrac{e^x}{(\ln x)^2}$

14) $f(x) = \dfrac{\ln(\sin x)}{e^x}$

1.8 Finding Local Extrema

Derivatives can be used to find a function's **local extreme values**, its peaks and valleys. At its peaks and valleys, a function's derivative is either 0 (a smooth, rounded peak/valley) or undefined (a sharp, pointy peak/valley).



Critical Points

The points at which a function's derivative is 0 or undefined, and the function itself exists, are called **critical points** of the function. We can find the critical points by taking the derivative, noting any singularities, setting the derivative to 0, and solving.

For example, to find the critical points of the function $f(x)=x\sqrt{1-x^2}$, we start by taking the derivative and simplifying.

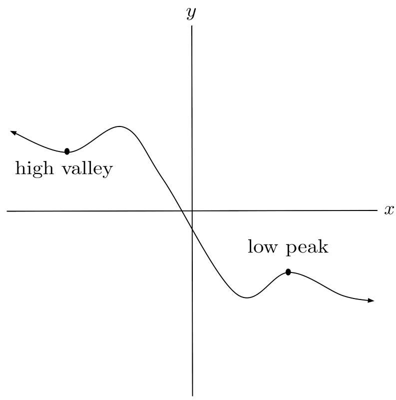
$$\begin{aligned} f(x) &= x\sqrt{1-x^2} \\ f'(x) &= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \\ &= \frac{1-2x^2}{\sqrt{1-x^2}} \end{aligned}$$

The derivative has a singularity when the denominator $\sqrt{1-x^2}$ is 0, which happens at $x=\pm 1$. The derivative itself is zero when the numerator $1-2x^2$ is 0, which happens at $x=\pm\sqrt{\frac{1}{2}}$. The function is defined at all of these x-values, so they all correspond to critical points: $x=\pm 1, \pm\sqrt{\frac{1}{2}}$.

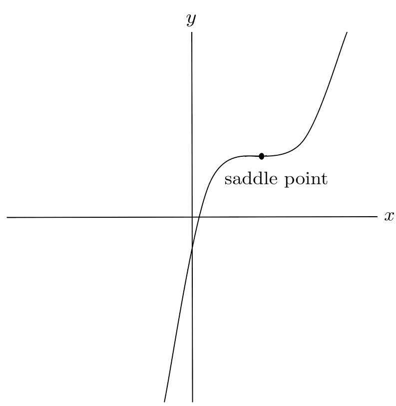
Classifying Critical Points

Now, how do we tell which critical points correspond to maxima (peaks), and which correspond to minima (valleys)?

It may be tempting to decide whether a critical point is a maximum or minimum by observing whether the resulting function value is large or small. However, it is entirely possible that some local minima may be greater than some local maxima. Think of a mountain range -- some valleys may be higher than some peaks.



It may also be possible that some critical points are neither peaks nor valleys, but **saddle points** on the side of a mountain where the terrain is flat. At saddle points like the one indicated below, the derivative is 0 but the point is neither a maximum nor a minimum.



First Derivative Test

There are two main methods for determining whether a critical point is a local minimum, local maximum, or neither. One way is to inspect the sign of the derivative on either side of the critical point, which tells whether we are ascending or descending on either side of the critical point.

- If the derivative is positive to the left of the critical point and negative to the right of the critical point, then we are ascending to a peak and then descending down the peak, which tells us that the critical point is a local maximum.

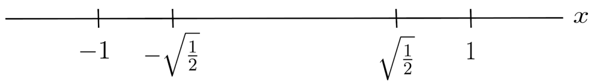
- On the other hand, if the derivative is negative to the left of the critical point and positive to the right of the critical point, then we are descending down a valley and then climbing up the valley, which tells us that the critical point is a local minimum.

- Lastly, if the derivative does not switch sign from the left of the critical point to the right of the critical point, then we are either ascending up the whole way or descending down the whole way, which indicates that the critical point is a saddle point.

This method is called the **first derivative test**, because it makes use of the first derivative of the function.

Demonstration of First Derivative Test

To use the first derivative test on the critical points $x = \pm 1, \pm\sqrt{\frac{1}{2}}$ that we found for the function $f(x) = x\sqrt{1-x^2}$, we first split up the number line over the critical points.



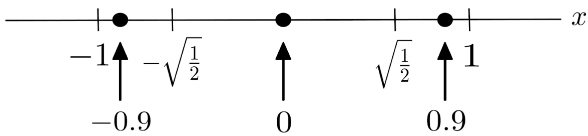
The number line splits into 5 intervals:

$$(-\infty, -1) \cup \left(-1, -\sqrt{\frac{1}{2}}\right) \cup \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) \cup \left(\sqrt{\frac{1}{2}}, 1\right) \cup (1, \infty)$$

However, on the intervals $(-\infty, -1)$ and $(1, \infty)$ our function $f(x) = x\sqrt{1-x^2}$ is not defined because the argument of the square root becomes negative. We remove these intervals from consideration.

$$\left(-1, -\sqrt{\frac{1}{2}}\right) \cup \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) \cup \left(\sqrt{\frac{1}{2}}, 1\right)$$

We want to know whether our function is increasing or decreasing on each of these intervals. To find out this information, we choose a test value in each of the remaining intervals. The actual values of the test values don't matter, because the derivative maintains the same sign within any given interval. For the sake of example, we choose our test values as, say, -0.9, 0, and 0.9.



Lastly, we evaluate the sign of the derivative at each of these test values.

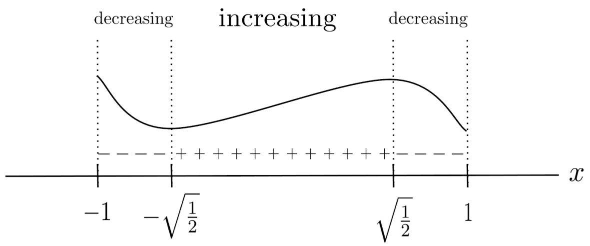
$$f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}}$$

$$f'(-0.9) = \frac{1-2(-0.9)^2}{\sqrt{1-(-0.9)^2}} = \frac{-}{\sqrt{+}} = -$$

$$f'(0) = \frac{1-2(0)^2}{\sqrt{1-(0)^2}} = \frac{+}{\sqrt{+}} = +$$

$$f'(0.9) = \frac{1-2(0.9)^2}{\sqrt{1-(0.9)^2}} = \frac{-}{\sqrt{+}} = -$$

The sign of the derivative at each particular test value tells us the sign of the derivative throughout the interval containing the particular test value. As a result, we know whether the function is increasing or decreasing on each interval, and we can sketch a rough graph of the peaks and valleys of the function.



We see that the function $f(x) = x\sqrt{1-x^2}$ has maxima at $x = -1, \sqrt{\frac{1}{2}}$ and minima at $x = -\sqrt{\frac{1}{2}}, 1$.

Second Derivative Test

The other method for classifying a critical point of a function as a maximum or minimum is called the **second derivative test**, because it makes use of the second derivative of the function.

- If the second derivative is positive at the critical point, then the function is concave up in the shape of a smile, which means the critical point is a local minimum.

- If the second derivative is negative at the critical point, then the function is concave down in the shape of a frown, which means the critical point is a local maximum.

- If the second derivative is 0 or undefined at the critical point, then we cannot conclude whether the critical point is a local maximum or minimum, and we need to fall back to the first derivative test.

The second derivative test is sometimes inconclusive, but it is mentioned because it is often faster than the first derivative test.

Demonstration of Second Derivative Test

To use the second derivative test on the critical points $x = \pm 1, \pm\sqrt{\frac{1}{2}}$ which we found for the function $f(x) = x\sqrt{1-x^2}$, we first take the second derivative of the function. We computed the first derivative earlier, so we just have to differentiate once more.

$$\begin{aligned}
f'(x) &= \frac{1-2x^2}{\sqrt{1-x^2}} \\
f''(x) &= \frac{-4x\sqrt{1-x^2} + \frac{x(1-2x^2)}{\sqrt{1-x^2}}}{1-x^2} \\
&= \frac{-4x(1-x^2) + x(1-2x^2)}{(1-x^2)^{\frac{3}{2}}} \\
&= \frac{2x^3 - 3x}{(1-x^2)^{\frac{3}{2}}}
\end{aligned}$$

We evaluate the sign of the second derivative at each of the critical points.

$$f''(-1) = \frac{-2+3}{(1-1)^{\frac{3}{2}}} = \frac{+}{0} = \text{undefined}$$

$$f''\left(-\sqrt{\frac{1}{2}}\right) = \frac{\sqrt{\frac{1}{2}} + 3\sqrt{\frac{1}{2}}}{\left(1-\frac{1}{2}\right)^{\frac{3}{2}}} = \frac{+}{+} = +$$

$$f''\left(\sqrt{\frac{1}{2}}\right) = \frac{\sqrt{\frac{1}{2}} - 3\sqrt{\frac{1}{2}}}{\left(1-\frac{1}{2}\right)^{\frac{3}{2}}} = \frac{-}{+} = -$$

$$f''(1) = \frac{2-3}{(1-1)^{\frac{3}{2}}} = \frac{-}{0} = \text{undefined}$$

Based on the results of the second derivative test, we see that $x = -\sqrt{\frac{1}{2}}$ is a minimum, and $x = \sqrt{\frac{1}{2}}$ is a maximum. The test is inconclusive for $x = -1$ and $x = 1$, so we would need to fall back to the first derivative test for these cases.

When to Use Each Test

In general, it's a good idea to use the first derivative test when the second derivative is more complex than the first derivative, and the second derivative test when the second derivative is less complex than the first derivative.

For example, for polynomial functions, it is usually easiest to use the second derivative test because the second derivative is less complex than the first derivative.

$$\begin{aligned} p(x) &= 2x^3 - 3x^2 - 11x + 6 \\ p'(x) &= 6x^2 - 6x - 11 \\ p''(x) &= 12x - 6 \end{aligned}$$

We find the critical points by solving for where the first derivative is zero.

$$\begin{aligned} 0 &= 6x^2 - 6x - 11 \\ x &= \frac{3 \pm 5\sqrt{3}}{6} \end{aligned}$$

Then, we find the sign of the second derivative at these points.

$$\begin{aligned} p''\left(\frac{3-5\sqrt{3}}{6}\right) &= 12\left(\frac{3-5\sqrt{3}}{6}\right) - 6 = 2(3-5\sqrt{3}) - 6 = -10\sqrt{3} < 0 \\ p''\left(\frac{3+5\sqrt{3}}{6}\right) &= 12\left(\frac{3+5\sqrt{3}}{6}\right) - 6 = 2(3+5\sqrt{3}) - 6 = 10\sqrt{3} > 0 \end{aligned}$$

The critical point $x = \frac{3-5\sqrt{3}}{6}$ has a negative second derivative, which means the function is concave down and thus the critical point is a maximum. Likewise, the critical point $x = \frac{3+5\sqrt{3}}{6}$ has a positive second derivative, which means the function is concave up and thus the critical point is a minimum.

On the other hand, for the function below, it is easiest to use the first derivative test because the computations for the second derivative will get a bit messy when we use the product rule.

$$\begin{aligned} f(x) &= e^{x^2+\frac{1}{x}} \\ f'(x) &= \left(2x - \frac{1}{x^2}\right) e^{x^2+\frac{1}{x}} \end{aligned}$$

We find the critical points by solving for where the first derivative is zero.

$$\begin{aligned} 0 &= \left(2x - \frac{1}{x^2}\right) e^{x^2+\frac{1}{x}} \\ 0 &= 2x - \frac{1}{x^2} \\ 1 &= 2x^3 \\ \sqrt[3]{\frac{1}{2}} &= x \end{aligned}$$

We choose test points $x = -1$ and $x = 1$ on each side of our critical point, and evaluate the sign of the first derivative at these points.

$$\begin{aligned} f'(-1) &= (-2-1)e^{1-1} = (-)(+) = - \\ f'(1) &= (2-1)e^{1+1} = (+)(+) = + \end{aligned}$$

The function has a negative derivative to the left of the critical point and a positive derivative to the right of the critical point, which means it is descending to the critical point and then ascending from the critical point. Therefore, the critical point $x = \sqrt[3]{\frac{1}{2}}$ is a minimum of the function.

Functions Defined on Closed Intervals

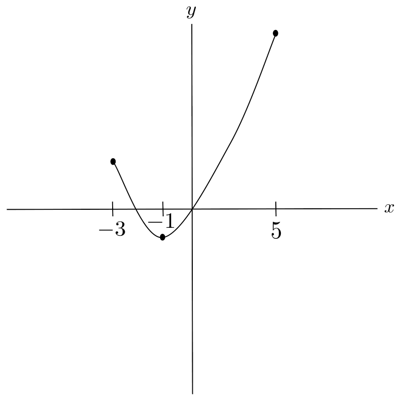
Lastly, when a function is defined on a closed interval, we need to use the endpoints as critical points as well, because the derivative isn't defined there but the function is.

For example, to find the extrema of the function $f(x) = x^2 + 2x$ with $x \in [-3, 5]$, we should also consider $x = -3$ and $x = 5$ as critical points, in addition to the point $x = -1$ which makes the derivative $f'(x) = 2x + 2$ equal to zero.

To apply the first derivative test, we choose a test point $x = -2$ for the interval $[-3, -1)$ and $x = 0$ for the interval $(-1, 5]$.

$$\begin{aligned} f'(-2) &= 2(-2) + 2 = - \\ f'(0) &= 2(0) + 2 = + \end{aligned}$$

The function is decreasing from $x = -3$ to $x = -1$, and then increasing from $x = -1$ to $x = 5$. Therefore, the function has a minimum at $x = -1$ and maxima at $x = -3$ and $x = 5$.



Exercises

For each function, find the critical points and label each critical point as a local maximum, local minimum, or saddle point.

1) $f(x) = x^3 - 2x + 1$

2) $f(x) = x^4 + 10x^3 - 5$

3) $f(x) = x \ln x$

4) $f(x) = \ln(5x - x^2)$

5) $f(x) = \dfrac{x^2 + 1}{x + 1}$

6) $f(x) = \dfrac{1}{x^2} e^{x^2 - 1}$

7) $f(x) = 4 - x^3$
$x \in [-1, 2]$

8) $f(x) = x^3 - 4x^2$
$x \in [-3, 3]$

9) $f(x) = \dfrac{x^2}{x + 1}$
$x \in [-5, 5]$

10) $f(x) = \dfrac{1 + \ln x}{x}$
$x \in \left[\dfrac{1}{2}, 10\right]$

1.9 Differentials and Approximation

The chain rule tells us that we can treat the derivative $\frac{df}{dx}$ like a fraction when multiplying by other derivatives. In this chapter, we continue the idea of interpreting the derivative as a fraction, extending it to an even more literal sense.

The main idea of **differentials** is that we can interpret the derivative $\frac{df}{dx}$ as an approximation for how the function output changes, when the function input is changed by a small amount. The terms df and dx are called **differentials**, and we can interpret them as small changes in the function's output and input.

Demonstration

For example, if we know that $f(4) = 2$ and $f'(4) = 5$ for some function $f(x)$, then we can estimate the value of $f(4.1)$ by treating the differentials as small changes in x and $f(x)$.

$$\begin{aligned}
\left.\frac{df}{dx}\right|_{x=4} &= 5 \\
\left.\frac{\Delta f}{\Delta x}\right|_{x=4} &\approx 5 \\
\frac{f(4.1)-f(4)}{4.1-4} &\approx 5 \\
\frac{f(4.1)-2}{0.1} &\approx 5 \\
f(4.1)-2 &\approx 0.5 \\
f(4.1) &\approx 2.5
\end{aligned}$$

Estimating Trig Functions and Roots

We can use this method to estimate values of functions that are difficult to compute, like trig functions and roots.

For example, we know that $\sin 0 = 0$ and that $(\sin x)'\big|_{x=0} = \cos 0 = 1$, so we can estimate the value of $\sin 0.1$ using differentials.

$$\begin{aligned}
\left.\frac{d(\sin x)}{dx}\right|_{x=0} &\approx \left.\frac{\Delta(\sin x)}{\Delta x}\right|_{x=0} \\
\cos 0 &\approx \frac{\sin 0.1 - \sin 0}{0.1 - 0} \\
1 &\approx \frac{\sin 0.1 - 0}{0.1} \\
0.1 &\approx \sin 0.1
\end{aligned}$$

Our estimation is pretty good -- the actual value of $\sin 0.1$ is $0.099833\ldots$

Similarly, we know that $\sqrt{144} = 12$ and that $(\sqrt{x})'\big|_{x=144} = \frac{1}{2\sqrt{144}} = \frac{1}{24}$, so we can estimate the value of $\sqrt{149}$ using differentials.

$$\begin{aligned} \frac{d(\sqrt{x})}{dx}\bigg|_{x=144} &\approx \frac{\Delta(\sqrt{x})}{\Delta x}\bigg|_{x=144} \\ \frac{1}{2\sqrt{144}} &\approx \frac{\sqrt{149}-\sqrt{144}}{149-144} \\ \frac{1}{24} &\approx \frac{\sqrt{149}-12}{5} \\ \frac{5}{24} &\approx \sqrt{147}-12 \\ 12\frac{5}{24} &\approx \sqrt{147} \\ 12.208 &\approx \sqrt{147} \end{aligned}$$

Again, our estimation is pretty good -- the actual value of $\sqrt{147}$ is $12.206555\ldots$

Intuition

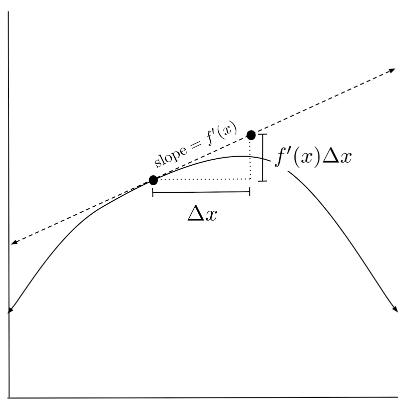
To understand why we can interpret the differentials as small changes, remember that the difference quotient is a good approximation for the derivative, when the difference Δx is small.

$$\begin{aligned}\frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &\approx \frac{f(x+\Delta x) - f(x)}{\Delta x}\end{aligned}$$

The numerator is really just the change in the values of the function f, so we can represent it by $\Delta f = f(x+\Delta x) - f(x)$.

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x}$$

Graphically, approximating via differentials amounts to approximating with a tangent line. We start at the point $(x, f(x))$, travel Δx units horizontally, and find the y-value that allows us to maintain a slope of $f'(x)$.



Since the tangent line goes through the point $(x, f(x))$ with slope $f'(x)$, the points (X, Y) on the tangent line are given by the following linear equation in point-slope form:

$$Y - f(x) = f'(x)(X - x)$$

Interpreting $X = x + \Delta x$ and $Y \approx f(x + \Delta x)$, we see that this equation is equivalent to the one we've been working with.

$$\begin{aligned}
Y - f(x) &= f'(x)(X - x) \\
\frac{Y - f(x)}{X - x} &= f'(x) \\
\frac{Y - f(x)}{x + \Delta x - x} &= f'(x) \\
\frac{Y - f(x)}{\Delta x} &= f'(x) \\
\frac{f(x + \Delta x) - f(x)}{\Delta x} &\approx f'(x) \\
\frac{\Delta f}{\Delta x} &\approx f'(x)
\end{aligned}$$

Exercises

Approximate each value by using differentials and the given equality. In your computations, use $\pi \approx 3.14$, $e \approx 2.72$, and $\sqrt{3} \approx 1.73$, and round to 2 decimal places.

1) $(4.8)^2 \approx$ ___
$5^2 = 25$

2) $\sqrt{15} \approx$ ___
$\sqrt{16} = 4$

3) $\sqrt[3]{-7} \approx$ ___
$\sqrt[3]{-8} = -2$

4) $\cos\left(\dfrac{7\pi}{30}\right) \approx$ ___
$\cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

5) $\sin\left(\dfrac{11\pi}{30}\right) \approx$ ___
$\sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

6) $\tan\left(\dfrac{5\pi}{16}\right) \approx$ ___
$\tan\left(\dfrac{\pi}{4}\right) = 1$

7) $\ln 3 \approx$ ___
$\ln 2.72 \approx 1$

8) $e^{1.1} \approx$ ___
$e^1 \approx 2.72$

9) $\arcsin 0.47 \approx$ ___
$\arcsin 0.5 = \dfrac{\pi}{6}$

10) $\arccos 0.54 \approx$ ___
$\arccos 0.5 = \dfrac{\pi}{3}$

1.10 L'Hôpital's Rule

L'Hôpital's rule provides a way to evaluate limits that take the indeterminate forms of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. It says that, for such limits, we can differentiate the numerator and denominator separately, without changing the actual value of the limit.

For example, the following limit has indeterminate form.

$$\lim_{x \to 1} \frac{\ln x}{1-x} = \frac{0}{0}$$

Therefore, we can apply L'Hôpital's rule to solve it.

$$\begin{aligned}\lim_{x \to 1} \frac{\ln x}{1-x} &= \lim_{x \to 1} \frac{(\ln x)'}{(1-x)'} \\ &= \lim_{x \to 1} \frac{\frac{1}{x}}{-1} \\ &= \lim_{x \to 1} -\frac{1}{x} \\ &= -1\end{aligned}$$

Products in Indeterminate Form

Limits of the form $0 \cdot \infty$ are also indeterminate, but we need to convert them to an equivalent fraction before applying L'Hôpital's rule.

For example, the following limit has indeterminate form of $0 \cdot \infty$, so we convert it to an equivalent fraction which has indeterminate form $\frac{\infty}{\infty}$.

$$\begin{aligned}
\lim_{x \to 0^+} \sqrt{x} \ln x &= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \frac{-\infty}{\infty} \\
&= \lim_{x \to 0^+} \frac{(\ln x)'}{\left(\frac{1}{\sqrt{x}}\right)'} \\
&= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}} \\
&= \lim_{x \to 0^+} -2\sqrt{x} \\
&= 0
\end{aligned}$$

We could use other equivalent fractions, too, as long as they are equivalent to the original limit and have indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

However, even though L'Hôpital's rule applies to any fraction having indeterminate form, some fractions are better than others. For example, if we wrote the previous limit as

$$\lim_{x \to 0^+} \sqrt{x} \ln x = \lim_{x \to 0^+} \frac{\sqrt{x}}{\frac{1}{\ln x}}$$

we would still have indeterminate form and thus be able to apply L'Hôpital's rule, but we wouldn't get anywhere with it because the derivative of $\frac{1}{\ln x}$ gets more complex. The point of using L'Hôpital's rule is to use differentiation to reduce the complexity of the limit, not increase it.

$$\begin{aligned}
\lim_{x\to 0^+} \sqrt{x}\ln x &= \lim_{x\to 0^+} \frac{\sqrt{x}}{\frac{1}{\ln x}} = \frac{0}{0} \\
&= \lim_{x\to 0^+} \frac{\left(\sqrt{x}\right)'}{\left(\frac{1}{\ln x}\right)'} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{2\sqrt{x}}}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} \\
&= \lim_{x\to 0^+} -\frac{1}{2}(\ln x)^2\sqrt{x}
\end{aligned}$$

Combining L’Hôpital’s Rule with Other Methods

Sometimes, we may have to use other methods in conjunction with L’Hôpital’s rule. For example, to solve the limit

$$\lim_{x\to 0^+} \left(\frac{1}{x}\right)^x$$

we can first compute the logarithm of the limit, using L’Hôpital’s rule.

$$\begin{aligned}
\ln \lim_{x\to 0^+} \left(\frac{1}{x}\right)^x &= \lim_{x\to 0^+} \ln\left[\left(\frac{1}{x}\right)^x\right] \\
&= \lim_{x\to 0^+} x\ln\left(\frac{1}{x}\right) \\
&= \lim_{x\to 0^+} -x\ln x \\
&= \lim_{x\to 0^+} -\frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty} \\
&= \lim_{x\to 0^+} -\frac{(\ln x)'}{\left(\frac{1}{x}\right)'} \\
&= \lim_{x\to 0^+} -\frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0^+} x \\
&= 0
\end{aligned}$$

So, we have:

$$\lim_{x\to 0^+} \left(\frac{1}{x}\right)^x = e^0 = 1$$

Limits that are Not in Indeterminate Form

One BIG word of caution: L'Hôpital's rule does NOT apply when a limit does not have indeterminate form. If you try to use L'Hôpital's rule on a limit that does not have indeterminate form, then it may lead you to an erroneous result.

For example, the limit $\lim_{x\to\infty} \frac{\sin x}{x}$, does not take indeterminate form since the numerator does not go to zero nor infinity, and we know using the squeeze theorem that the limit evaluates to 0. But if we apply L'Hôpital's rule on this limit, we conclude that the limit does not exist, which is incorrect since it actually does exist and evaluates to 0.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sin x}{x} &= \lim_{x\to\infty} \frac{\cos x}{1} && \text{(invalid)} \\
&= \lim_{x\to\infty} \cos x && \text{(invalid)} \\
&= \text{does not exist} && \text{(invalid)}
\end{aligned}$$

Derivation and Mean Value Theorem

To see why L'Hôpital's rule works, we can start off noticing that the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

implies that $f(a) = 0$ and $g(a) = 0$. This is obvious, but it's very important to notice, because it lets us express the above limit as the ratio of difference quotients.

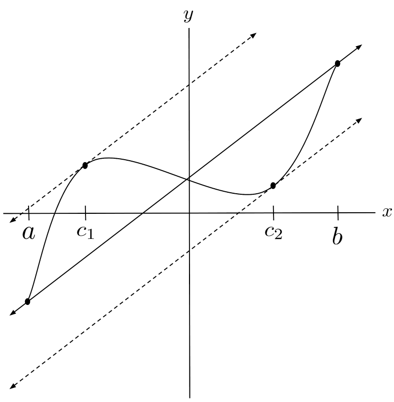
$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{\Delta x\to 0}\frac{f(a+\Delta x)}{g(a+\Delta x)} \\
&= \lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{g(a+\Delta x)-g(a)} \\
&= \lim_{\Delta x\to 0}\frac{\frac{f(a+\Delta x)-f(a)}{\Delta x}}{\frac{g(a+\Delta x)-g(a)}{\Delta x}} \\
&= \frac{\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}}{\lim_{\Delta x\to 0}\frac{g(a+\Delta x)-g(a)}{\Delta x}} \\
&= \frac{f'(a)}{g'(a)}
\end{aligned}$$

This is pretty close to the full statement of L'Hôpital's rule, but it is a bit more limited because it assumes that $g'(a)$ is nonzero -- it assumes that the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ can be evaluated through direct substitution, to yield $\frac{f'(a)}{g'(a)}$. But we have broken these assumptions in some examples, where we applied L'Hôpital's rule multiple times in a row -- in these examples, the limit still couldn't be evaluated by direct substitution after a single iteration of L'Hôpital's rule.

To overcome these assumptions and prove the full statement of L'Hôpital's rule we need to understand the **mean value theorem**, which says that for any function $f(x)$ that is continuous on an interval $[a,b]$ and differentiable on the interval (a,b), there is some point $x=c$ at which the derivative of f is equal to its average rate of change:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

In other words, the mean value theorem says that if we draw a line between the endpoints of f, it will be parallel to the tangent line of f somewhere in the interval.



When fiddling with the mean value theorem, you might notice that the mean value theorem is a particular case of a more general and elegant equation, with $g(x) = x$.

$$\frac{f'(c)}{1} = \frac{f(b) - f(a)}{b - a}$$
$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

To check whether this extended result is true for any function $g(x)$, we can ask whether the derivative of the following function $h(x)$ is 0 at some point $c \in [a, b]$.

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$$

Interestingly, this function has the property $h(a) = h(b)$, so the mean value theorem tells us that, indeed, as long as $h(x)$ is continuous on an interval $[a, b]$ and differentiable on the interval (a, b), then it is true that $h'(c) = 0$ for some point $c \in [a, b]$. And the assumptions of continuity and differentiability are true for $h(x)$ whenever they are true for $f(x)$ and $g(x)$, so the mean value theorem does in fact extend to the result

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

This result is known, rather intuitively, as the extended mean value theorem.

L'Hôpital's rule comes from applying the extended mean value theorem to the limit in question. If we have the indeterminate limit

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \frac{0}{0}$$

then we consider the interval $[a, x]$. Here, both $f(a) = 0$ and $g(a) = 0$, and the extended mean value theorem tells us that for some $c \in [a, x]$ we have the following:

$$\begin{aligned}\frac{f'(c)}{g'(c)} &= \frac{f(x)-f(a)}{g(x)-g(a)} \\ &= \frac{f(x)-0}{g(x)-0} \\ &= \frac{f(x)}{g(x)}\end{aligned}$$

Taking the limit as $x \to a^+$, we have $c \to a^+$ and thus

$$\begin{aligned}\lim_{x\to a^+}\frac{f'(c)}{g'(c)} &= \lim_{x\to a^+}\frac{f(x)}{g(x)} \\ \lim_{x\to a^+}\frac{f'(x)}{g'(x)} &= \lim_{x\to a^+}\frac{f(x)}{g(x)}.\end{aligned}$$

The indeterminate limit

$$\lim_{x\to a^-}\frac{f(x)}{g(x)} = \frac{0}{0}$$

can be understood the same way using the interval $[x, a]$, and the indeterminate limits

$$\begin{aligned}\lim_{x\to \infty}\frac{f(x)}{g(x)} &= \frac{0}{0} \\ \lim_{x\to -\infty}\frac{f(x)}{g(x)} &= \frac{0}{0}\end{aligned}$$

can be understood similarly, using the intervals $[x, \infty)$ and $(-\infty, x]$. Likewise, in the case of

$$\lim \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

we can rewrite the limit as

$$\lim \frac{\frac{1}{f(x)}}{\frac{1}{g(x)}} = \frac{0}{0}$$

and apply L'Hôpital's rule, which ends up simplifying to its original form.

$$\begin{aligned}
\lim \frac{\frac{1}{f(x)}}{\frac{1}{g(x)}} &= \lim \frac{\frac{1}{f(x)}}{\frac{1}{g(x)}} \\
\lim \frac{\frac{1}{f(x)}}{\frac{1}{g(x)}} &= \lim \frac{\left(\frac{1}{f(x)}\right)'}{\left(\frac{1}{g(x)}\right)'} \\
\lim \frac{\frac{1}{f(x)}}{\frac{1}{g(x)}} &= \lim \frac{-\frac{f'(x)}{f(x)^2}}{-\frac{g'(x)}{g(x)^2}} \\
\lim \frac{\frac{1}{f(x)}}{\frac{1}{g(x)}} &= \lim \frac{f'(x)}{g'(x)} \cdot \frac{g(x)^2}{f(x)^2} \\
\lim \frac{g(x)}{f(x)} &= \lim \frac{f'(x)}{g'(x)} \cdot \frac{g(x)^2}{f(x)^2} \\
\lim \frac{g(x)}{f(x)} \cdot \frac{f(x)^2}{g(x)^2} &= \lim \frac{f'(x)}{g'(x)} \\
\lim \frac{f(x)}{g(x)} &= \lim \frac{f'(x)}{g'(x)}
\end{aligned}$$

Exercises

Evaluate the indicated limits by applying L'Hôpital's rule.

1) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

2) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$

3) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\tan x}{\frac{\pi}{2}-x}$

4) $\lim\limits_{x\to 0^+} \dfrac{(\ln x)^2}{x}$

5) $\lim\limits_{x\to 1} \dfrac{x\ln x}{x^2-1}$

6) $\lim\limits_{x\to 0^+} (\ln x)\tan x$

7) $\lim\limits_{x\to \frac{\pi}{2}} (\tan x)^{\cos x}$

8) $\lim\limits_{x\to \infty} x^{\frac{1}{\sqrt{x}}}$

9) $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

10) $\lim\limits_{x\to \infty} x^{\frac{1}{\ln x}}$

Part 2
Integrals

2.1 Antiderivatives

An **antiderivative** of a function $f(x)$ is a function $F(x)$ whose derivative is $f(x)$, i.e. $F'(x) = f(x)$.

For example, an antiderivative of $2x$ is x^2, because $(x^2)' = 2x$. Another antiderivative of $2x$ is $x^2 + 1$, because $(x^2 + 1)' = 2x$. To encapsulate all possibilities, we say that the antiderivative of $2x$ is $x^2 + C$ where C is a constant.

The antiderivative of a function $f(x)$ is written symbolically as $\int f(x)\, dx$. For example, to say that the antiderivative of $2x$ is $x^2 + C$, we can write $\int 2x\, dx = x^2 + C$.

The symbol $\int$ is called an **integral**, and the differential dx tells us that x is the variable of integration. (The variable of integration may seem unnecessary right now, but it will become more relevant in later chapters when we talk about techniques to solve integrals.)

Power Rule

The power rule for differentiation tells us that $(x^n)' = nx^{n-1}$. Through a bit of clever intuition, we find a function whose derivative is x^n.

$$\left(\frac{1}{n+1}x^{n+1}\right)' = x^n$$

Consequently, we have a power rule for integration:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

A few examples are shown below.

$$\int x^2 \, dx = \frac{1}{3} x^3 + C$$

$$\begin{aligned} \int \frac{1}{x^5} \, dx &= \int x^{-5} \, dx \\ &= \frac{1}{-4} x^{-4} + C \\ &= -\frac{1}{4x^4} + C \end{aligned}$$

$$\begin{aligned} \int \sqrt{x} \, dx &= \int x^{\frac{1}{2}} \, dx \\ &= \frac{1}{3/2} x^{\frac{3}{2}} + C \\ &= \frac{2}{3} x^{\frac{3}{2}} + C \end{aligned}$$

$$\begin{aligned} \int 1 \, dx &= \int x^0 \, dx \\ &= \frac{1}{1} x^1 + C \\ &= x + C \end{aligned}$$

Integral of the Reciprocal Function

You might notice that if we try to use this power rule to integrate $\frac{1}{x}$, which simplifies to x^{-1}, we come up with a nonsense result.

$$\begin{aligned}\int \frac{1}{x}\,dx &= \int x^{-1}\,dx \\ &= \frac{1}{-1+1}x^0 + C \qquad \text{(invalid)} \\ &= \frac{1}{0} + C \qquad \text{(invalid)}\end{aligned}$$

The case of $n = -1$ is an exception to the power rule, and if we try to perform the power rule anyway, we obtain an invalid result. We will see in a later chapter that, surprisingly, the antiderivative of $\frac{1}{x}$ is $\ln x$.

$$\int \frac{1}{x}\,dx = \ln x + C$$

Sum and Constant Multiple Rules

Integrals exhibit some of the same properties as derivatives. For example, the integral of a sum can be computed as the sum of integrals of the integral terms. Also, constants can be moved outside of integrals.

$$\begin{aligned}
&\int 3x^3 + x^2 - 4x + 2\,dx \\
&= \int 3x^3\,dx + \int x^2\,dx + \int -4x\,dx + \int 2\,dx + C \\
&= 3\int x^3\,dx + \int x^2\,dx - 4\int x\,dx + 2\int 1\,dx + C \\
&= 3\left(\frac{1}{4}x^4\right) + \frac{1}{3}x^3 - 4\left(\frac{1}{2}x^2\right) + 2(x) + C \\
&= \frac{3}{4}x^4 + \frac{1}{3}x^3 - 2x^2 + 2x + C
\end{aligned}$$

Note that although each individual integral in the sum is associated with a $+C$ constant term, they are redundant, because when we combine all the terms together we still get a constant. Thus, we are able to write a single $+C$ at the very end to account for all constants that arise from the multiple individual integrals.

Integrating Products and Quotients

Unfortunately, there is no simple rule for integrating a product or quotient. We will learn techniques later to make such integrals easier, but for now, the best strategy is to expand out the function as much as possible before trying to take the integral.

For example, to integrate the product $(x+1)(x-1)(x^2+2)$, we can multiply out the product and then integrate each term individually.

$$\begin{aligned}
&\int (x+1)(x-1)(x^2+2)\,dx \\
&= \int (x^2-1)(x^2+2)\,dx \\
&= \int x^4+x^2-2\,dx \\
&= \frac{1}{5}x^5+\frac{1}{3}x^3-2x+C
\end{aligned}$$

Similarly, to integrate the quotient $\frac{2x^3+x^2-4}{x^2}$, we can split up each term in the numerator and then simplify.

$$\begin{aligned}
&\int \frac{2x^3+x^2-4}{x^2}\,dx \\
&= \int \frac{2x^3}{x^2}+\frac{x^2}{x^2}-\frac{4}{x^2}\,dx \\
&= \int 2x+1-4x^{-2}\,dx \\
&= x^2+x+4x^{-1}+C \\
&= x^2+x+\frac{4}{x}+C
\end{aligned}$$

Integrating Non-Polynomial Functions

Below are some useful rules for finding antiderivatives of non-polynomial functions. (For the sake of readability, the dx and $+C$ have been removed.)

$$\int x^n = \frac{1}{n+1}x^{n-1}\ (n \neq 1) \qquad \int \frac{1}{x} = \ln x$$

$$\int e^{nx} = \frac{1}{n}e^{nx} \qquad \int n^x = \frac{1}{\ln n}n^x$$

$$\int \cos(nx) = \frac{1}{n}\sin(nx) \qquad \int \sin(nx) = -\frac{1}{n}\cos(nx)$$

$$\int \sec^2(nx) = \frac{1}{n}\tan(nx) \qquad \int \csc^2(nx) = -\frac{1}{n}\cot(nx)$$

$$\int \sec(nx)\tan(nx) = \frac{1}{n}\sec(nx)$$

$$\int \csc(nx)\cot(nx) = -\frac{1}{n}\csc(nx)$$

$$\int \frac{1}{\sqrt{1-(nx)^2}} = \frac{1}{n}\arcsin(nx)$$

$$\int -\frac{1}{\sqrt{1-(nx)^2}} = \frac{1}{n}\arccos(nx)$$

$$\int \frac{1}{1+(nx)^2} = \frac{1}{n}\arctan(nx)$$

Non-polynomial functions can be integrated similarly: we simplify the integral as much as we can, and then find the antiderivative of each term separately. A few examples are shown below.

$$\begin{aligned}
&\int (\sin 3x)\left(2+\frac{1}{\cos^2(5x)\sin(3x)}\right)\,dx \\
&= \int 2\sin 3x + \frac{1}{\cos^2 5x}\,dx \\
&= \int 2\sin 3x + \sec^2 5x\,dx \\
&= 2\int \sin 3x\,dx + \int \sec^2 5x\,dx \\
&= 2\left(-\frac{1}{3}\cos 3x\right) + \frac{1}{5}\tan 5x + C \\
&= -\frac{2}{3}\cos 3x + \frac{1}{5}\tan 5x + C
\end{aligned}$$

$$\begin{aligned}
&\int \frac{(e^{4x}+3)(e^{4x}-3)}{e^{5x}} + 7\cdot 2^x + 3^{-x}\,dx \\
&= \int \frac{e^{8x}-9}{e^{5x}} + 7\cdot 2^x + \left(\frac{1}{3}\right)^x\,dx \\
&= \int e^{3x} - 9e^{-5x} + 7\cdot 2^x + \left(\frac{1}{3}\right)^x\,dx \\
&= \int e^{3x}\,dx - 9\int e^{-5x}\,dx + 7\int 2^x\,dx + \int \left(\frac{1}{3}\right)^x\,dx \\
&= \frac{1}{3}e^{3x} - 9\left(\frac{1}{-5}e^{-5x}\right) + 7\left(\frac{1}{\ln 2}2^x\right) + \frac{1}{\ln\frac{1}{3}}\left(\frac{1}{3}\right)^x + C \\
&= \frac{1}{3}e^{3x} + \frac{9}{5}e^{-5x} + \frac{7}{\ln 2}2^x + \frac{1}{-\ln 3}3^{-x} + C \\
&= \frac{1}{3}e^{3x} + \frac{9}{5}e^{-5x} + \frac{7}{\ln 2}2^x - \frac{1}{\ln 3}3^{-x} + C
\end{aligned}$$

$$\begin{aligned}
&\int \frac{1}{3+2x^2} - \frac{1}{\sqrt{1-4x^2}}\,dx \\
&= \int \frac{1}{3}\left(\frac{1}{1+\frac{2}{3}x^2}\right) - \frac{1}{\sqrt{1-(2x)^2}}\,dx \\
&= \int \frac{1}{3}\left(\frac{1}{1+\left(\sqrt{\frac{2}{3}}x\right)^2}\right) - \frac{1}{\sqrt{1-(2x)^2}}\,dx \\
&= \frac{1}{3}\int \frac{1}{1+\left(\sqrt{\frac{2}{3}}x\right)^2}\,dx - \int \frac{1}{\sqrt{1-(2x)^2}}\,dx \\
&= \frac{1}{3}\left(\frac{1}{\sqrt{\frac{2}{3}}}\arctan\sqrt{\frac{2}{3}}x\right) - \frac{1}{2}\arcsin 2x + C \\
&= \frac{1}{\sqrt{6}}\arctan\sqrt{\frac{2}{3}}x - \frac{1}{2}\arcsin 2x + C
\end{aligned}$$

Exercises

Evaluate the following integrals.

1) $\displaystyle\int x^3 - 3x^2 + \frac{6}{x^4}\,dx$

2) $\displaystyle\int x\left(8x^2 - \frac{1}{x}\right)dx$

3) $\displaystyle\int (x^2+2)^2\,dx$

4) $\displaystyle\int \frac{(x+1)(2x+3)}{3x^2}\,dx$

5) $\displaystyle\int 2\sec^2 3x + \csc\frac{x}{2}\cot\frac{x}{2}\,dx$

6) $\displaystyle\int \frac{\sin 3x}{\cos^2 3x}\,dx$

7) $\displaystyle\int \left(\sin\frac{1}{4}x\right)\left(3 - \frac{4}{\cos^2 5x \sin\frac{1}{4}x}\right)dx$

8) $\displaystyle\int \frac{\cos(\pi x) - 10}{3\sin^2(\pi x)}\,dx$

9) $\displaystyle\int e^{4x} - 3e^{-3x} + e^{-x}\,dx$

10) $\displaystyle\int (2e^{3x} - 1)(e^{-5x} + 2)\,dx$

11) $\displaystyle\int \frac{(2e^{4x} - e^{2x})(e^x + 1)}{e^{4x}}\,dx$

12) $\displaystyle\int \frac{\sqrt{e^{3x}} + 2}{e^{3x}}\,dx$

13) $\displaystyle\int \frac{1}{4 + 4x^2} - \frac{5}{\sqrt{1 - x^2}}\,dx$

14) $\displaystyle\int \frac{2}{1 + x^2} + \frac{1}{\sqrt{3 - 3x^2}}\,dx$

15) $\displaystyle\int \frac{1}{1 + 9x^2} - \frac{3}{\sqrt{1 - 16x^2}}\,dx$

16) $$\int \frac{1}{5 - 10x^2} + \frac{1}{\sqrt{4 + 9x^2}}\,dx$$

2.2 Finding Area

In the last chapter, we learned how to evaluate integrals of the form $\int f(x)\,dx$, which are also known as **indefinite integrals**. In this chapter, we shall be concerned with **definite integrals**, which have bounds of integration.

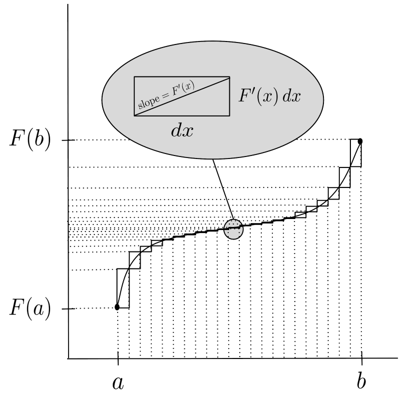
The definite integral $\int_a^b f(x)\,dx$ is evaluated by first finding the antiderivative $F(x) = \int f(x)\,dx$, and then computing the difference between the values of the antiderivative at the indicated bounds.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

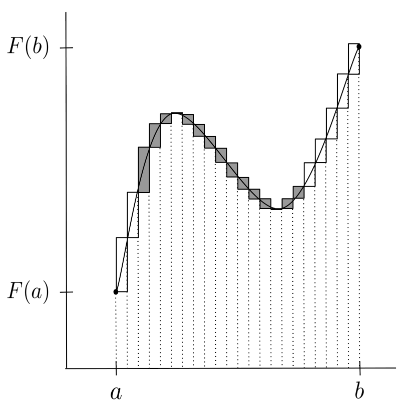
Derivation

Subtracting at the bounds yields the area between the x-axis and the function $f(x)$, between the bounds $x = a$ and $x = b$. To see why, first consider that $F(b) - F(a)$ is the sum of infinitely many, infinitely small changes in F, one for each value of x. At each value x, the function has slope $F'(x)$, so if it travels an infinitesimal dx units to the right, then it also travels an infinitesimal $F'(x)\,dx$ units up.

$$F(b) - F(a) = \sum_{x \in [a,b]} F'(x)\,dx$$



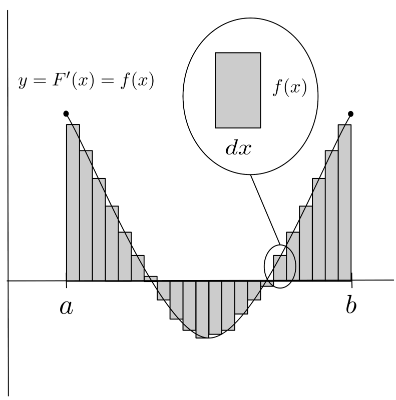
This is true even if the function doubles back on itself, because the upward and downward displacements cancel each other out.



Because $F(x) = \int f(x)\,dx$, we have $F'(x) = f(x)$, so we can write the sum in terms of f.

$$\begin{aligned} F(b) - F(a) &= \sum_{x \in [a,b]} F'(x)\,dx \\ &= \sum_{x \in [a,b]} f(x)\,dx \end{aligned}$$

Each term in the sum then corresponds to the area of a rectangle of width dx and height $f(x)$, and all the rectangles together make up the area between the x-axis and the graph of f.

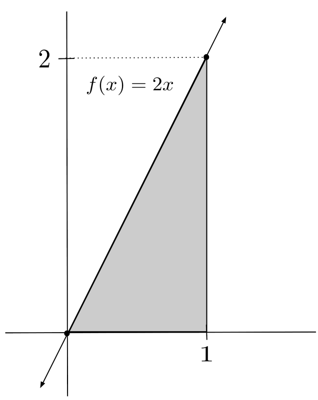


Sanity Check

Below is an example of evaluating a simple definite integral.

$$\begin{aligned}\int_0^1 2x\,dx &= \left[x^2\right]_0^1 \\ &= 1^2 - 0^2 \\ &= 1 - 0 \\ &= 1\end{aligned}$$

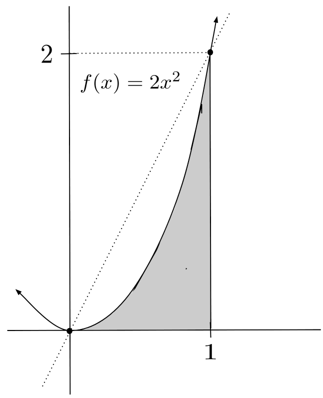
We can verify that this result represents the area between the x-axis and the function $f(x) = 2x$ between the bounds $x = 0$ and $x = 1$, because this region is just a triangle. The results match up!



$$\begin{aligned}\text{area of triangle} &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(1)(2) \\ &= 1\end{aligned}$$

Demonstration

Now, let's compute the area between the x-axis and the parabola $f(x) = 2x^2$, between the same bounds $x = 0$ and $x = 1$. The parabola dips a little lower than the triangle which we found has area 1, so we should expect a result a little smaller than 1.



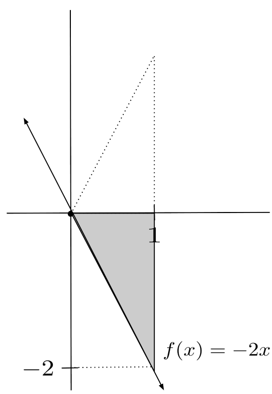
$$\begin{aligned} \int_0^1 2x^2\,dx &= \left[\frac{2}{3}x^3\right]_0^1 \\ &= \frac{2}{3}(1)^3 - \frac{2}{3}(0)^3 \\ &= \frac{2}{3} - 0 \\ &= \frac{2}{3} \end{aligned}$$

The area is $\frac{2}{3}$, which is indeed slightly smaller than 1, so it matches up with our expectations.

Negative Area

When a function dips below the x-axis, the area below the x-axis is counted as negative area.

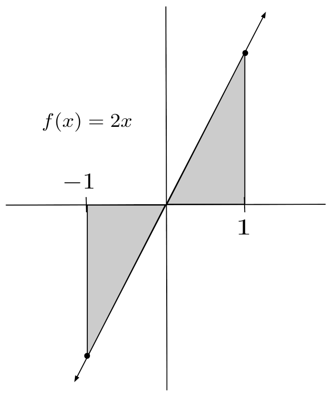
For example, if we integrate the function $f(x) = -2x$ between the bounds $x = 0$ and $x = 1$, we get a result of -1. This is the same triangle as before, but flipped over the x-axis.



$$\begin{aligned}\int_0^1 -2x\,dx &= \left[-x^2\right]_0^1 \\ &= -(1)^2 - (-(0)^2) \\ &= -1 + 0 \\ &= -1\end{aligned}$$

As a consequence of negative area, for a region that has the same amount of area above the x-axis as below the x-axis, the integral will evaluate to 0.

For example, the function $f(x) = 2x$ integrates to zero on the interval from $x = -1$ to $x = 1$ because its two triangles above and below the x-axis cancel each other out.



$$\begin{aligned}\int_{-1}^{1} 2x\,dx &= \left[x^2\right]_{-1}^{1} \\ &= 1^2 - (-1)^2 \\ &= 1 - 1 \\ &= 0\end{aligned}$$

Area Between Two Functions

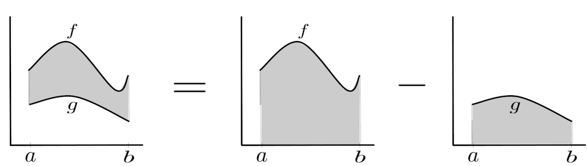
In addition to finding the area under a single function, integrals can also be used to find the area between two functions.

If we have two functions $f(x)$ and $g(x)$ with $f(x) \geq g(x)$ on the interval $[a,b]$, then the area between the functions on the interval $[a,b]$ is given by the integral of the difference:

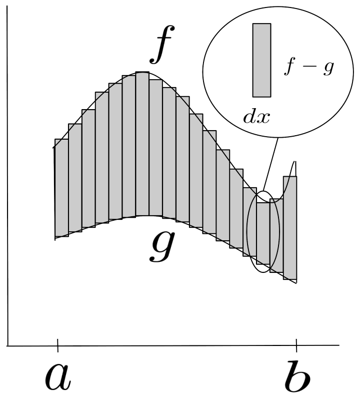
$$\text{area between } f \text{ and } g = \int_a^b f(x) - g(x)\,dx$$

One way to interpret the integral above is to see it as the difference of two separate integrals, the integral of f minus the integral of g. Then the area between f and g is the area under f minus the overlapping area under g, which leaves only the area between f and g.

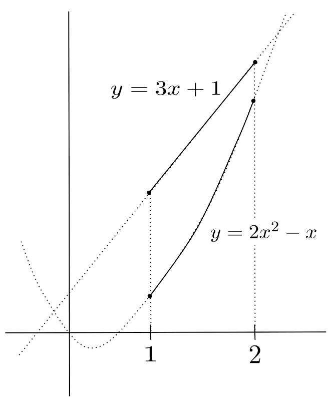
$$\begin{aligned}\int_a^b f(x) - g(x)\,dx &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= (\text{area under } f) - (\text{area under } g) \\ &= \text{area between } f \text{ and } g\end{aligned}$$



Another way to interpret the integral is to see it as integrating the height from g to f. In this case, we are defining a height function $h(x) = f(x) - g(x)$ and breaking the region between the functions into infinitesimally small rectangles, each rectangle having height $h(x)$ and infinitesimal width dx.



For example, to find the area between $y = 3x + 1$ and $y = 2x^2 - x$ on the interval $[1, 2]$, we first need to identify which function is the higher one on this interval. We can do this by sketching graphs of the functions.



We see that $y=3x+1$ is the higher function and $y=2x^2-x$ is the lower function. The integral is the higher function $3x+1$ minus the lower function $2x^2-x$, over the interval $[1,2]$.

$$\begin{aligned}
&\int_1^2 (3x+1)-(2x^2-x)\,dx \\
&= \int_1^2 3x+1-2x^2+x\,dx \\
&= \int_1^2 -2x^2+4x+1\,dx \\
&= \left[-\frac{2}{3}x^3+2x^2+x\right]_1^2 \\
&= \left[-\frac{2}{3}(2)^3+2(2)^2+(2)\right]-\left[-\frac{2}{3}(1)^3+2(1)^2+(1)\right] \\
&= \left[-\frac{16}{3}+8+2\right]-\left[-\frac{2}{3}+2+1\right] \\
&= \left[\frac{14}{3}\right]-\left[\frac{7}{3}\right] \\
&= \frac{7}{3}
\end{aligned}$$

So, the area between the functions $y=3x+1$ and $y=2x^2-x$ on the interval $[1,2]$ is $\frac{7}{3}$.

Area Between Two Functions that Intersect

Sometimes, two functions will cross on an interval, and each will take its turn being the higher function. For example, the functions $y = x$ and $y = x^2$ cross twice on the interval $[-1, 3]$. The points of intersection are obtained by setting the functions equal to each other and solving:

$$\begin{aligned} x &= x^2 \\ 0 &= x^2 - x \\ 0 &= x(x-1) \\ x &= 0, 1 \end{aligned}$$

On $[-1, 0]$ the higher function is $y = x^2$, on $[0, 1]$ the higher function is $y = x$, and on $[1, 3]$ the higher function is $y = x^2$. To find the total area bounded between the functions, we integrate the higher function minus the lower function on each interval and add the results together.

$$\begin{aligned} &\int_{-1}^{0} x^2 - x \, dx + \int_{0}^{1} x - x^2 \, dx + \int_{1}^{3} x^2 - x \, dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 \right]_{-1}^{0} + \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{0}^{1} + \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 \right]_{1}^{3} \\ &= \left[-\left(-\frac{1}{3} - \frac{1}{2} \right) \right] + \left[\frac{1}{2} - \frac{1}{3} \right] + \left[\left(9 - \frac{9}{2} \right) - \left(\frac{1}{3} - \frac{1}{2} \right) \right] \\ &= \frac{5}{6} + \frac{1}{6} + \frac{28}{6} \\ &= \frac{17}{3} \end{aligned}$$

To perform the computation faster, we can ignore which function is higher vs lower provided we take the absolute value of each integral before adding them together. Even if we end up "incorrectly" computing the lower function minus the higher function in some integral, the result will still represent area -- it will just be negative area, so we can correct it by making it positive.

If we treat $y = x^2$ as the higher function on all intervals but take the absolute value of the integrals before adding them, we reach the same result as before.

$$\begin{aligned}
&\left|\int_{-1}^{0} x^2 - x\,dx\right| + \left|\int_{0}^{1} x^2 - x\,dx\right| + \left|\int_{1}^{3} x^2 - x\,dx\right| \\
&= \left|\left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-1}^{0}\right| + \left|\left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{0}^{1}\right| + \left|\left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{1}^{3}\right| \\
&= \left|\frac{5}{6}\right| + \left|-\frac{1}{6}\right| + \left|\frac{28}{6}\right| \\
&= \frac{5}{6} + \frac{1}{6} + \frac{28}{6} \\
&= \frac{17}{3}
\end{aligned}$$

Likewise, if we treat $y = x$ as the higher function on all intervals but take the absolute value of the integrals before adding them, we reach the same result as before.

$$\begin{aligned}
&\left|\int_{-1}^{0} x - x^2\,dx\right| + \left|\int_{0}^{1} x - x^2\,dx\right| + \left|\int_{1}^{3} x - x^2\,dx\right| \\
&= \left|\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{0}\right| + \left|\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{0}^{1}\right| + \left|\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{1}^{3}\right| \\
&= \left|-\frac{5}{6}\right| + \left|\frac{1}{6}\right| + \left|-\frac{28}{6}\right| \\
&= \frac{5}{6} + \frac{1}{6} + \frac{28}{6} \\
&= \frac{17}{3}
\end{aligned}$$

Exercises

Find the net (signed) area below each function on the given interval.

1) $y = 12x^3 + x$
$x \in [-1, 2]$

2) $y = \sin x - \sec^2 x$
$x \in \left[0, \frac{\pi}{3}\right]$

3) $y = 3e^{5x}$
$x \in [-2, 0]$

4) $y = \sqrt{x} + \sin x$
$x \in [\pi, 9\pi]$

Find the area between the two functions on the given interval.

5) $y = 2^x$
$y = e^x$
$x \in [0, 1]$

6) $y = x - 1$
$y = x^2$
$x \in [-2, 2]$

7) $y = \sqrt{x}$
$y = \ln x$
$x \in [e, e^2]$

8) $y = \cos x$
$y = 4 - x^2$
$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

9) $y = x + 1$
$y = x^2 - 1$
$x \in [1, 3]$

10) $y = \sin x$
$y = \cos x$
$x \in [0, \pi]$

11) $y = x^2$
$y = \sqrt[3]{x}$
$x \in [-1, 8]$

12) $y = 1 - x^2$
$y = x^2 - 1$
$x \in [-2, 2]$

2.3 Substitution

Complicated integrals can sometimes be made simpler through the method of **substitution**. Substitution involves condensing an expression of x into a single variable, say u, and then expressing the integral in terms of u instead of x.

Demonstration

To make the idea of substitution more concrete, consider the integral $\int (3x+1)^8\,dx$. We may be tempted to use the power rule, and say that the integral evaluates to $\frac{1}{9}(3x+1)^9$. But if we differentiate to check our result, we see that, because of the chain rule, the derivative of this expression is not equal to the function inside the integral.

$$\begin{aligned}\left[\frac{1}{9}(3x+1)^9\right]' &= \frac{1}{9}\left[(3x+1)^9\right]' \\ &= \frac{1}{9}\cdot 9(3x+1)^8(3x+1)' \\ &= (3x+1)^8(3) \\ &= 3(3x+1)^8 \\ &\neq (3x+1)^8\end{aligned}$$

To turn the integral into one that can be solved with the power rule, we condense the $3x+1$ expression into a single variable u, through the substitution $u=3x+1$.

$$\int (3x+1)^8 \, dx = \int u^8 \, dx$$

Before we apply the power rule, we need to take care of one issue: the differential is still dx, and we need it to be du. In general, we can't just replace the dx differential with a du differential. However, by interpreting the derivative as a fraction, we can solve for the dx differential in terms of the du differential.

$$\begin{aligned} \frac{du}{dx} &= (3x+1)' \\ \frac{du}{dx} &= 3 \\ du &= 3dx \\ \frac{1}{3}du &= dx \\ dx &= \frac{1}{3}du \end{aligned}$$

Once our integral is fully expressed in terms of u, we can solve it via the power rule, and then substitute $u=3x+1$ again to write our answer in terms of x.

$$\begin{aligned}\int u^8\,dx &= \int u^8 \cdot \frac{1}{3}\,du \\ &= \frac{1}{3}\int u^8\,du \\ &= \frac{1}{3}\cdot\frac{1}{9}u^9 + C \\ &= \frac{1}{27}u^9 + C \\ &= \frac{1}{27}(3x+1)^9 + C\end{aligned}$$

We verify that the derivative of the result is indeed the original function within the integral.

$$\begin{aligned}\left[\frac{1}{27}(3x+1)^9\right]' &= \frac{1}{27}\left[(3x+1)^9\right]' \\ &= \frac{1}{27}\cdot 9(3x+1)^8(3x+1)' \\ &= \frac{1}{3}(3x+1)^8(3) \\ &= (3x+1)^8\end{aligned}$$

Choosing the Right Substitution

The key to substitution is choosing the right substitution. But how can we tell what is the right substitution? For example, in the integral below, should we substitute $u = \sin x$ or $u = \cos x$?

$$\int \sin^2 x \cos x\,dx$$

Whenever we are torn between multiple substitution choices, we should choose the substitution whose derivative will cancel out other terms in the integral.

In this case, we should choose $u = \sin x$, because the derivative $u' = \cos x$ will cancel out the existing $\cos x$ inside the integral. On the other hand, $u = \cos x$ would not work, because the derivative $u' = -\sin x$ would not fully cancel the existing $\sin^2 x$ inside the integral.

Choosing $u = \sin x$, we have $\frac{du}{dx} = \cos x$, so $dx = \frac{1}{\cos x} du$. Substituting into the integral, we are able to evaluate.

$$\begin{aligned} \int \sin^2 x \cos x \, dx &= \int u^2 \cos x \cdot \frac{1}{\cos x} du \\ &= \int u^2 du \\ &= \frac{1}{3} u^3 + C \\ &= \frac{1}{3} \sin^3 x + C \end{aligned}$$

Exercises

Evaluate each integral using substitution.

1) $\int \sqrt{x+2}\, dx$

2) $\int (4x+3)^8 \, dx$

3) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

4) $\displaystyle\int \frac{3x^2}{(x^3-5)^3}\,dx$

5) $\displaystyle\int \sec^2 x \tan^2 x\,dx$

6) $\displaystyle\int \frac{\cos x}{\sqrt{\sin x}}\,dx$

7) $\displaystyle\int \frac{\sec^2 x \tan x}{(\sec^2 x+1)^4}\,dx$

8) $\displaystyle\int \cos(\cos x)\sin x\,dx$

9) $\displaystyle\int x^2 e^{x^3+1}\,dx$

10) $\displaystyle\int \frac{\csc^2 x}{e^{\cot x}}\,dx$

11) $\displaystyle\int \frac{x^2 e^{\sqrt{x^3-1}}}{\sqrt{x^3-1}}\,dx$

12) $\displaystyle\int e^{x+e^x}\,dx$

13) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

14) $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

15) $\displaystyle\int \frac{1}{\sqrt{x-x^2}}\,dx$

16) $\displaystyle\int \frac{x}{x^4+1}\,dx$

2.4 Integration by Parts

Integration by parts is another technique for simplifying integrals. We can apply integration by parts whenever an integral would be made simpler by differentiating some expression within the integral, at the cost of anti-differentiating another expression within the integral. The formula for integration by parts is given below:

$$\int u\,dv = uv - \int v\,du$$

The formula is really just a direct consequence of the product rule -- we can obtain it by applying the product rule to a product uv, integrating with respect to x, and rearranging a bit.

$$\begin{aligned}
\frac{d}{dx}[uv] &= u\frac{dv}{dx} + v\frac{du}{dx} \\
\int \frac{d}{dx}[uv]\;dx &= \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx \\
uv &= \int u\,dv + \int v\,du \\
\int u\,dv &= uv - \int v\,du
\end{aligned}$$

Demonstration

To see why integration by parts is useful, consider the integral $\int xe^x \, dx$. If we differentiate the x term, then the term goes away, and if we integrate the e^x term, the term stays the same. Therefore, by applying integration by parts, we can simplify the integral.

We choose $u = x$ and $dv = e^x \, dx$. Since $u = x$, we have $\frac{du}{dx} = 1$, so $du = dx$. Since $dv = e^x \, dx$, we have $v = \int e^x \, dx = e^x$. (We ignore the constant of integration now because we're saving it for the very end.) Substituting this information into the integration by parts formula, we are able to evaluate the integral.

$$\begin{aligned} \int xe^x \, dx &= xe^x - \int 1e^x \, dx \\ &= xe^x - \int e^x \, dx \\ &= xe^x - e^x + C \\ &= (x-1)e^x + C \end{aligned}$$

Repeated Application

Sometimes, we may have to perform integration by parts more than once.

For example, in the following integral, the first integration by parts reduces the x^2 to $2x$, and the second integration by parts reduces the $2x$ to 2, which finally simplifies the integral to a point where we can solve it.

$$\int x^2 \sin x \, dx$$

To start off, we choose $u = x^2$ and $dv = \sin x \, dx$. Then $du = 2x \, dx$ and $v = -\cos x$, and the integral simplifies a bit.

$$\begin{aligned} &\int x^2 \sin x \, dx \\ &= x^2(-\cos x) - \int (-\cos x) 2x \, dx \\ &= -x^2 \cos x + \int 2x \cos x \, dx \end{aligned}$$

For the final round of integration by parts, we choose $u = 2x$ and $dv = \cos x \, dx$. Then $du = 2 \, dx$ and $v = \sin x$, and the integral simplifies a bit more, to a point where we can solve it.

$$\begin{aligned} &= -x^2 \cos x + \left(2x \sin x - \int 2 \sin x \, dx \right) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C \\ &= 2x \sin x + (2 - x^2) \cos x + C \end{aligned}$$

Cyclic Cases

Other times, integration by parts will never simplify an integral to a point where it can be directly computed.

For example, in the integral

$$\int e^x \cos x \, dx$$

differentiating the e^x term will not reduce its complexity because it just stays e^x, and differentiating the $\cos x$ term will not reduce its complexity because it just flips back and forth between $\sin x$ and $\cos x$.

However, we can use integration by parts to set up a recurrence equation, which can be used to solve algebraically for the integral. Choosing $u = \cos x$ and $dv = e^x \, dx$ we have $du = -\sin x \, dx$ and $v = e^x$.

$$\begin{aligned} &\int e^x \cos x \, dx \\ &= e^x \cos x - \int -e^x \sin x \, dx \\ &= e^x \cos x + \int e^x \sin x \, dx \end{aligned}$$

We perform one more round of integration by parts with $u = \sin x$ and $dv = e^x \, dx$, so that we have $du = \cos x \, dx$ and $v = e^x$.

$$\begin{aligned} &= e^x \cos x + \left(e^x \sin x - \int e^x \cos x \, dx \right) \\ &= (\sin x + \cos x) e^x - \int e^x \cos x \, dx \end{aligned}$$

Now that the original integral has reappeared in our expression, we can solve for it algebraically.

$$\begin{aligned} \int e^x \cos x \, dx &= (\sin x + \cos x)e^x - \int e^x \cos x \, dx \\ 2\int e^x \cos x \, dx &= (\sin x + \cos x)e^x \\ \int e^x \cos x \, dx &= \frac{1}{2}(\sin x + \cos x)e^x \end{aligned}$$

Then, since the integral is an indefinite integral, we just need to add a constant at the end.

$$\int e^x \cos x \, dx = \frac{1}{2}(\sin x + \cos x)e^x + C$$

Exercises

Use integration by parts to compute the following integrals.

1) $\int x^2 e^x \, dx$

2) $\int x \ln x \, dx$

3) $\int (x+1)\cos x \, dx$

4) $\int (2x^2 - 3x)e^x \, dx$

5) $\int x^5 e^{x^3} \, dx$

6) $\int e^x \sin x \, dx$

7) $\displaystyle\int (x \ln x)^2 \; dx$

8) $\displaystyle\int e^{2x} \sin\left(e^x\right) \; dx$

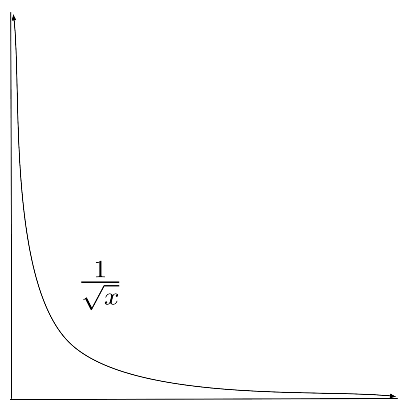
9) $\displaystyle\int \arctan\left(\frac{1}{x}\right) \; dx$

10) $\displaystyle\int \sin(2x)\cos(3x)\, dx$

2.5 Improper Integrals

Improper integrals have bounds or function values that extend to positive or negative infinity.

For example, $\int_1^\infty \frac{1}{x^2}\,dx$ is an improper integral because its upper bound is at infinity. Likewise, $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is an improper integral because $\frac{1}{\sqrt{x}}$ approaches infinity as x approaches the lower bound of integration, 0.



Convergence

It seems intuitive that improper integrals should always come out to infinity, since an infinitely long or infinitely high function would seemingly have infinite area.

However, although this can sometimes happen, it is not always the case. In fact, both of the two improper integrals given as examples in the previous paragraph evaluate to normal, non-infinite results. As such, we say that these integrals **converge**.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \left[-\frac{1}{x}\right]_1^\infty \\ &= -\frac{1}{\infty} - \left(-\frac{1}{1}\right) \\ &= 0+1 \\ &= 1\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \left[2\sqrt{x}\right]_0^1 \\ &= 2\sqrt{1} - 2\sqrt{0} \\ &= 2-0 \\ &= 2\end{aligned}$$

If the function decreases quickly enough as it extends out to infinity, then the area underneath it can come out to a finite number. Likewise, if a function blows up to infinity slowly enough as it approaches an asymptote, then the area underneath it can come out to a finite number.

Divergence

Below, we integrate the function $f(x) = \frac{1}{x}$, which decreases more slowly as it extends out to infinity and blows up to infinity more quickly as it approaches its vertical asymptote $x = 0$. The integrals of this function do indeed integrate to infinity. As such, we say that these integrals **diverge**.

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= [\ln x]_1^\infty \\ &= \ln\infty - \ln 1 \\ &= \infty - 0 \\ &= \infty\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{1}{x}\,dx &= [\ln x]_0^1 \\ &= \ln 1 - \ln 0 \\ &= 0 - (-\infty) \\ &= \infty\end{aligned}$$

Discontinuities within the Interval of Integration

Sometimes, a function may blow up to infinity somewhere within the interval of integration, rather than at the bounds of integration. In such a case, we have to separate the integral across its discontinuities.

For example, to compute the integral $\int_{-1}^2 \frac{1}{x^2}\,dx$, we may be tempted to ignore the singularity at $x = 0$ and simply evaluate the antiderivative at the bounds. This leads us to an invalid result.

$$\begin{aligned}
\int_{-1}^{2} \frac{1}{x^2}\,dx &= \left[-\frac{1}{x}\right]_{-1}^{2} && \text{(invalid)} \\
&= -\frac{1}{2} - \left(-\frac{1}{-1}\right) && \text{(invalid)} \\
&= -\frac{1}{2} - 1 && \text{(invalid)} \\
&= -\frac{3}{2} && \text{(invalid)}
\end{aligned}$$

This result of negative area doesn't make any sense, because the function $\frac{1}{x^2}$ is always positive!

In order to properly evaluate the integral $\int_{-1}^{2} \frac{1}{x^2}\,dx$, we have to split it up across the singularity, into two separate integrals.

The first integral spans from $x = -1$ to $x = 0$ and consequently approaches 0 from the negative side, so its computations involve 0^-.

The second integral spans from $x = 0$ to $x = 2$ and consequently approaches 0 from the positive side, so its computations involve 0^+.

$$\begin{aligned}
\int_{-1}^{2} \frac{1}{x^2}\,dx &= \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{2} \frac{1}{x^2}\,dx \\
&= \left[-\frac{1}{x}\right]_{-1}^{0^-} + \left[-\frac{1}{x}\right]_{0^+}^{2} \\
&= \left[-\frac{1}{0^-} - \left(-\frac{1}{-1}\right)\right] + \left[-\frac{1}{2} - \left(-\frac{1}{0^+}\right)\right] \\
&= [-(-\infty) - 1] + \left[-\frac{1}{2} - (-\infty)\right] \\
&= (\infty - 1) + \left(-\frac{1}{2} + \infty\right) \\
&= \infty + \infty \\
&= \infty
\end{aligned}$$

Now, we see that the integral actually diverges to infinity. This makes much more sense, since we know that it represents a region that contains a portion of infinite area.

Lastly, below is an example of a more complicated integral that converges.

$$\begin{aligned}
\int_{0}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx &= \left[-2e^{-\sqrt{x}}\right]_{0}^{\infty} \\
&= -2e^{-\infty} - \left(-2e^{0}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}$$

Exercises

Evaluate the improper integrals below.

1) $\displaystyle\int_1^\infty \frac{1}{x^5}\,dx$

2) $\displaystyle\int_3^\infty \frac{1}{\sqrt[3]{x}}\,dx$

3) $\displaystyle\int_5^\infty \frac{1}{x^2}\,dx$

4) $\displaystyle\int_{-\infty}^{-2} \frac{1}{(x-1)^4}\,dx$

5) $\displaystyle\int_0^1 \frac{1}{x-1}\,dx$

6) $\displaystyle\int_{-3}^3 \frac{1}{(x+2)^5}\,dx$

7) $\displaystyle\int_{\frac{3}{2}}^{10} \frac{1}{\sqrt{4x-6}}\,dx$

8) $\displaystyle\int_3^\infty \frac{1}{(2x+1)^{3/2}}\,dx$

9) $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^2}\,dx$

10) $\displaystyle\int_0^\infty \frac{1}{\left(\sqrt{x}+1\right)^2\sqrt{x}}\,dx$

11) $\displaystyle\int_0^\infty e^{-x}\,dx$

12) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

Part 3

Differential Equations

3.1 Separation of Variables

In differential equations, we are given an equation in terms of the derivative(s) of some function, and we need to solve for the function that makes the equation true.

For example, a simple differential equation is $y' = 2x$, and its solution is just the antiderivative $y = x^2 + C$.

$$\begin{aligned} y' &= 2x \\ (x^2 + C)' &= 2x \\ 2x &= 2x \end{aligned}$$

The simplest differential equations can be solved by **separation of variables**, in which we move the derivative to one side of the equation and take the antiderivative.

$$\begin{aligned} 3y' + \cos x &= 6x^2 + y' \\ 2y' + \cos x &= 6x^2 \\ 2y' &= 6x^2 - \cos x \\ y' &= 3x^2 - \frac{1}{2}\cos x \\ y &= \int 3x^2 - \frac{1}{2}\cos x \, dx \\ y &= x^3 - \frac{1}{2}\sin x + C \end{aligned}$$

Equations with a Higher-Order Derivative

This method can be used to solve simple equations with higher-order derivatives, as well.

$$
\begin{aligned}
y''' + x = 1 \\
y''' &= -x + 1 \\
y'' &= \int -x + 1 \, dx \\
y'' &= -\frac{1}{2}x^2 + x + C_1 \\
y' &= \int -\frac{1}{2}x^2 + x + C_1 \, dx \\
y' &= -\frac{1}{6}x^3 + \frac{1}{2}x^2 + C_1 x + C_2 \\
y &= \int -\frac{1}{6}x^3 + \frac{1}{2}x^2 + C_1 x + C_2 \, dx \\
y &= -\frac{1}{24}x^4 + \frac{1}{6}x^3 + C_1 x^2 + C_2 x + C_3
\end{aligned}
$$

Note that, although the antiderivative of $C_1 x$ is $\frac{C_1}{2}x^2$, the term $\frac{C_1}{2}$ is itself just a constant: $\frac{C_1}{2}x^2$ just means any constant multiplied by x^2. But $C_1 x^2$ also means any constant multiplied by x^2, so writing the fraction in $\frac{C_1}{2}x^2$ is redundant. To keep the notation simple and free of redundancy, we just write $C_1 x^2$.

Equations with Both Function and Derivative

When differential equations contain y terms as well as y' terms, we can still separate variables by using the differential notation for the derivative and treating it as a fraction.

$$\begin{aligned} y'y &= x \\ \frac{dy}{dx}y &= x \\ dy\,y &= x\,dx \\ y\,dy &= x\,dx \\ \int y\,dy &= \int x\,dx \\ \frac{1}{2}y^2 &= \frac{1}{2}x^2 + C \\ y^2 &= x^2 + C \\ y &= \pm\sqrt{x^2 + C} \end{aligned}$$

Even differential equations that contain two different variables multiplied together can sometimes be solved by separation of variables.

$$\begin{aligned}
y'e^y\cos^2 x &= 1\\
y'e^y &= \sec^2 x\\
\frac{dy}{dx}e^y &= \sec^2 x\\
dy\, e^y &= \sec^2 x\, dx\\
e^y\, dy &= \sec^2 x\, dx\\
\int e^y\, dy &= \int \sec^2 x\, dx\\
e^y &= \tan x + C\\
y &= \ln(\tan x + C)
\end{aligned}$$

But other times, there is no way to separate the variables from each other completely. We will learn more advanced methods to solve such non-separable differential equations in the coming chapters.

$$\begin{gathered}
y' + y = x\\
\frac{dy}{dx} + y = x\\
dy + y\, dx = x\, dx\\
\text{(unable to separate)}
\end{gathered}$$

Exercises

Solve the following differential equations using separation of variables.

1) $y' = 4$

2) $3y' = x^2$

3) $y' = x(1 + xy')$

4) $\left(\frac{y'}{x}\right)^3 = \sin^3(x^2)$

5) $y'y = \sin x$

6) $y'y = \frac{1}{x} - y'$

7) $(y+1)(1 - xe^x) = xy'$

8) $y' \cos y = x \sin y$

9) $y'' - 4e^{2x} = e^x$

10) $y''' + \cos x = x$

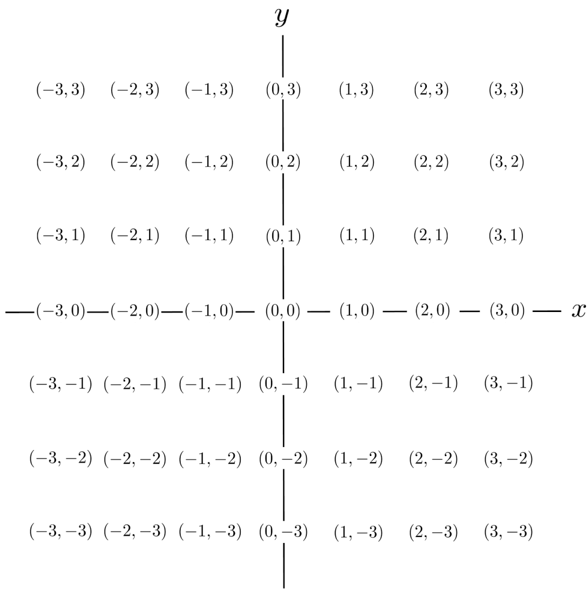
3.2 Slope Fields and Euler Approximation

When faced with a differential equation that we don't know how to solve, we can sometimes still approximate the solution by simpler methods. If we just want to get an idea of what the solutions of the differential equation look like on a graph, we can construct a **slope field**.

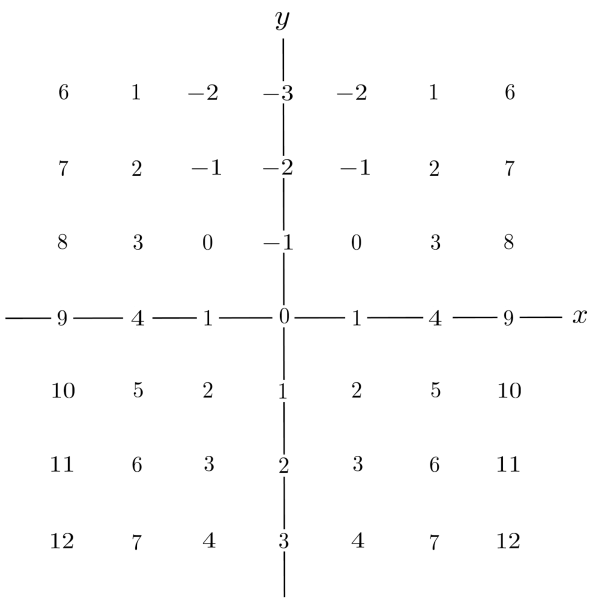
Slope Fields

A slope field consists of an array of line segments, each line segment angled so that it represents the slope at the corresponding point.

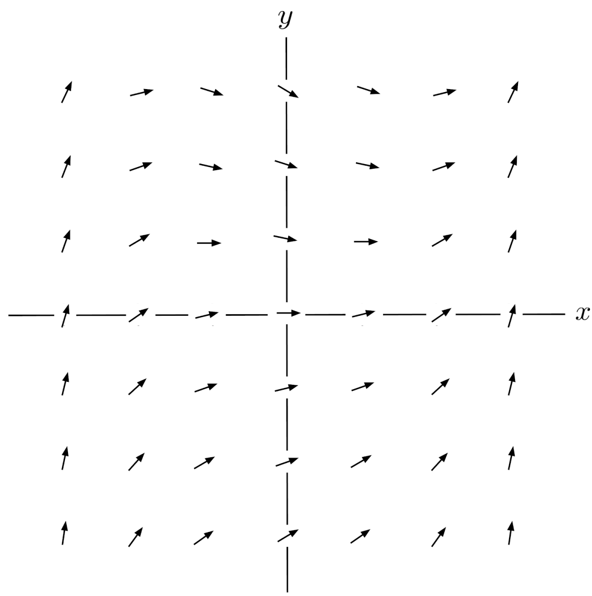
For example, to construct the slope field for the differential equation $y' = x^2 - y$, we start with an array of points.



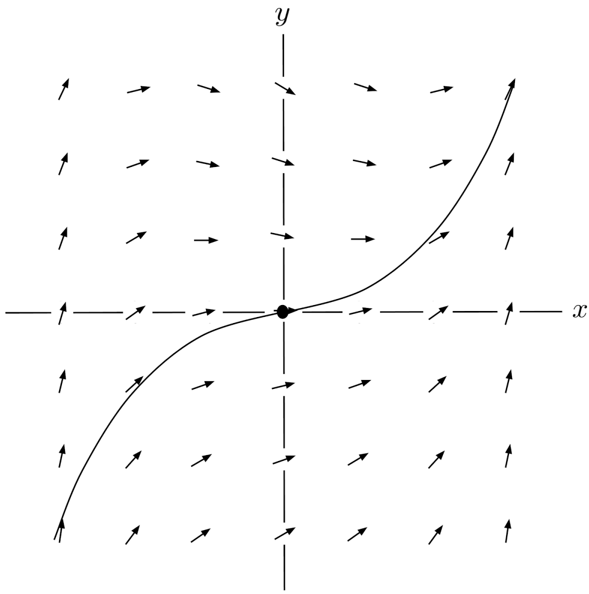
Then, we evaluate y' at each point (x, y).



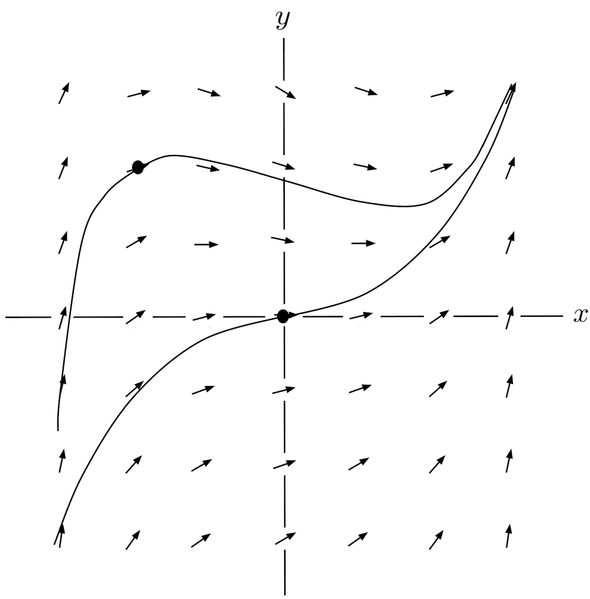
Lastly, we replace each value of y' with a short arrow having that slope.



Now, we have an idea of what the solutions of the differential equation look like. For example, if we start at the point $(0,0)$ and follow the slopes as we go left and right, then we end up with the following curve.



We can also choose a different point, say $(-2, 2)$ to see the solution curve which contains that point.



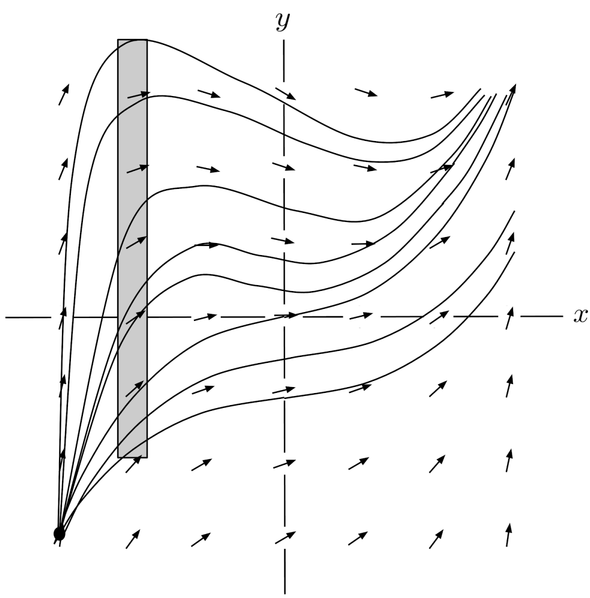
You can think of the coordinate plane as a river rapid, and the slope fields as the individual currents within the river rapid. If you launch a raft at a particular point, then the solution curve shows you where the river will take the raft.

Imprecision of Slope Fields

Although a slope field can show us the shapes of solutions to a differential equation, it isn't very precise.

For example, if a particular solution starts at the point $(-3,-3)$, then the slope field tells us that it travels up and right -- but exactly how far? If we travel right one units until the x-coordinate is -2, then what will the y-coordinate be?

Based on the sketch of the slope field, it's hard to tell whether the y-coordinate will be closer to -2 or 4. We need a more precise method.



Euler Estimation

We can estimate particular solutions more precisely using **Euler approximation**. In Euler approximation, we travel horizontally in small steps and use the derivative to compute how far we travel up or down at each step. The idea is that, since the solution curve is generated by this process with infinitesimally tiny step sizes, we can compute a good approximation to the solution curve if we use a small enough step size.

As an example, we will use Euler approximation to estimate the value of y when $x=-2$, starting from the point $(-3,-3)$. We will use a step size of $\Delta x = 0.25$.

We start by computing y' at the point $(-3,-3)$, using the differential equation $y' = x^2 - y$, and obtaining a result of $(-3)^2 - (-3) = 12$.

Then, using $\Delta x = 0.25$, we estimate Δy as $y'\Delta x$, which is $(0.25)(12) = 3$. We arrive at the point $(-3+0.25, -3+3)$, which simplifies to $(-2.75, 0)$.

At this point, we compute the derivative again, use it and Δx to estimate Δy, arrive at a new point, and continue the process until the x-coordinate is -2.

As shown in the table below, our resulting estimate of the y-coordinate is 3.5.

x	y	$y' = x^2 - y$	Δx	$\Delta y = y' \Delta x$
-3	-3	12	0.25	3
-2.75	0	7.56	0.25	1.89
-2.5	1.89	4.36	0.25	1.09
-2.25	2.98	2.08	0.25	0.52
$\mathbf{-2}$	$\mathbf{3.5}$			

Euler approximation tends to yield decent approximations for differential equations whose slope fields aren't too turbulent, and the approximations can be made more accurate by decreasing the step size.

However, for differential equations that have singularities, one must be careful applying Euler approximation because it can "step over" asymptotes.

Exercises

Draw slope fields for the following differential equations on the grid $-3 \leq x, y \leq 3$.

Then, sketch a rough graph of the solution that passes through the point $(0, 0)$.

Finally, starting at the point $(0, 0)$, use Euler estimation with 4

steps to approximate the value of y when $x = 1$. (Round to two decimal places throughout your calculations.)

1) $y' = y - x$

2) $y' = x^3 + y^3$

3) $y' = \sqrt{x^2 + y^2}$

4) $y' = \dfrac{1}{1 + |x + y|}$

3.3 Substitution

Sometimes, non-separable differential equations can be converted into separable differential equations by way of **substitution**.

For example, $y' + y = x$ is a non-separable differential equation as-is. However, we can make a variable substitution $u = x - y$ to turn it into a separable differential equation. Differentiating both sides of $u = x - y$ with respect to x, and interpreting y as a function of x, we have $u' = 1 - y'$, so $y' = 1 - u'$. Substituting, the equation becomes separable and thus solvable in terms of u.

$$\begin{aligned}
y' + y &= x \\
y' &= x - y \\
1 - u' &= u \\
u' &= 1 - u \\
\frac{du}{dx} &= 1 - u \\
du &= (1-u)\,dx \\
\frac{1}{1-u}\,du &= 1\,dx \\
\int \frac{1}{1-u}\,du &= \int 1\,dx \\
-\ln(1-u) &= x + C \\
\ln(1-u) &= -x + C \\
1 - u &= e^{-x+C} \\
u &= 1 - e^{-x+C}
\end{aligned}$$

Lastly, to find what y is, we can solve for y in our original substitution $u = x - y$.

$$\begin{aligned} u &= x - y \\ u + y &= x \\ y &= x - u \\ y &= x - \left(1 - e^{-x+C}\right) \\ y &= x - 1 + e^{-x+C} \end{aligned}$$

Choosing the Right Substitution

In general, to determine what substitution we need to perform, it is helpful to rearrange the equation until we see a group of terms whose derivative also appears in the equation.

$$\begin{aligned} 2yy' - y^2 &= x^2 - 2x \\ 2yy' + 2x &= x^2 + y^2 \\ (y^2 + x^2)' &= x^2 + y^2 \end{aligned}$$

After rearranging the above equation, we see that $u = x^2 + y^2$ is a good substitution. We rewrite the equation in terms of u, solve it, and then solve for y in terms of x.

$$\begin{aligned}
u' &= u \\
u &= Ce^x \\
x^2 + y^2 &= Ce^x \\
y^2 &= Ce^x - x^2 \\
y &= \pm\sqrt{Ce^x - x^2}
\end{aligned}$$

We don't always have to use addition in our substitutions. In the equation below, for example, we require the substitution $u = xy$.

$$\begin{aligned}
xy' &= 3 - y \\
xy' + y &= 3 \\
(xy)' &= 3 \\
u' &= 3 \\
u &= 3x + C \\
xy &= 3x + C \\
y &= 3 + \frac{C}{x}
\end{aligned}$$

=

Exercises

Use substitution to solve the following differential equations.

1) $1 + y' = (x+y)^2$

2) $2(y' - y) = 1 - x$

3) $x^2 - y^2 = \dfrac{1}{2x - 2yy'}$

4) $3y^2y' = e^{x^2+y^3} - 2x$

5) $2y + xy' = \dfrac{1}{x}$

6) $xy^4 + 2x^2y^3y' = 1$

3.4 Characteristic Polynomial

In this chapter, we learn a technique for solving differential equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constant coefficients, and $y^{(n)}$ denotes the n[th] derivative of y.

The **characteristic polynomial** of the differential equation above is given by

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0.$$

Each root r of the characteristic polynomial corresponds to a solution $(C_{r,1} + C_{r,2}x + C_{r,3}x^2 + \ldots + C_{r,m}x^{m-1})e^{rx}$ of the original equation, where m is the multiplicity of the root and $C_{r,1}, C_{r,2}, C_{r,3}, \ldots, C_{r,m}$ are unknown constants of integration.

The constants of integration are labeled intricately, each with two subscripts, so that we can stay organized, in case we have to deal with multiple roots.

Demonstration

For example, the differential equation $y'' - 3y' + 2y = 0$ has the characteristic polynomial $r^2 - 3r + 2$, which factors to $(r-2)(r-1)$ and has roots $r = 1, 2$.

The root $r = 1$ has multiplicity 1, which corresponds to a solution $C_{1,1}e^{1x}$ or more simply $C_{1,1}e^{x}$.

The root $r = 2$ also has multiplicity 1, which corresponds to a solution of $C_{2,1}e^{2x}$.

The full solution of the equation, then, is $y = C_{1,1}e^{x} + C_{2,1}e^{2x}$.

Another Demonstration

Next, consider the differential equation $y'' - 6y' + 9y = 0$.

This differential equation has the characteristic polynomial $r^2 - 6r + 9$, which factors to $(r-3)^2$ and has a single root $r = 3$ with multiplicity 2.

The solution of the equation, then, is $y = (C_{3,1} + C_{3,2}x)e^{3x}$.

Case of Imaginary Roots

Sometimes, the characteristic polynomial of a differential equation may have imaginary roots.

For example, the differential equation $y'' + 4y = 0$ has the characteristic polynomial $r^2 + 4$, which has roots $r = 2i, -2i$. In these cases, we apply the same procedure as before, but we take it a step further. We use **Euler's formula**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

to evaluate any exponentials with imaginary powers, and then we remove any i's from the solution. We can remove the i's because in general, if $f(x)i$ is a solution, then so is $f(x)$. This is true because the i can be factored out:

$$\begin{aligned} a_n(f(x)i)^{(n)} + a_{n-1}(f(x)i)^{(n-1)} + \cdots + a_2(f(x)i)'' + a_1(f(x)i)' + a_0 f(x)i = 0 \\ a_n f^{(n)}(x)i + a_{n-1}f^{(n-1)}(x)i + \cdots + a_2 f''(x)i + a_1 f'(x)i + a_0 f(x)i = 0 \\ \left[a_n f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \cdots + a_2 f''(x) + a_1 f'(x) + a_0 f(x)\right] i = 0 \\ a_n f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \cdots + a_2 f''(x) + a_1 f'(x) + a_0 f(x) = 0 \end{aligned}$$

Continuing the example, the root $r = 2i$ corresponds to a solution $C_{2i,1}e^{2ix}$, which simplifies to $C_{2i,1}(\cos 2x + i\sin 2x)$. Removing the i from this solution yields $C_{2i,1}(\cos 2x + \sin 2x)$.

By the same reasoning, the root $r = -2i$ corresponds to a solution $C_{-2i,1}(\cos(-2x) + \sin(-2x))$. Since $\cos(-\theta) = \cos\theta$ and

$\sin(-\theta) = -\sin\theta$ for all inputs θ, this solution simplifies further to $C_{-2i,1}(\cos 2x - \sin 2x)$.

The full solution, then, is

$$y = C_{2i,1}(\cos 2x + \sin 2x) + C_{-2i,1}(\cos 2x - \sin 2x)$$

which simplifies to

$$y = (C_{2i,1} + C_{-2i,1})\cos 2x + (C_{2i,1} - C_{-2i,1})\sin 2x.$$

It is redundant to use four constants in this solution, though, since $C_{2i,1} + C_{-2i,1}$ represents a single constant and $C_{2i,1} - C_{-2i,1}$ represents another single constant.

For example, if $C_{2i,1} = 1$ and $C_{-2i,1} = 2$, then the solution is just $3\cos 2x - \sin 2x$. We can make $C_{2i,1} + C_{-2i,1}$ and $C_{2i,1} - C_{-2i,1}$ come out to anything we want, by choosing $C_{2i,1}$ and $C_{-2i,1}$ accordingly.

Therefore, to avoid redundancy in the full solution, we replace the expression $C_{2i,1} + C_{-2i,1}$ with a single constant C_1, and the expression $C_{2i,1} - C_{-2i,1}$ with a single constant C_2.

$$y = C_1 \cos 2x + C_2 \sin 2x$$

Case of Complex Roots

When the characteristic polynomial has complex roots, the solutions will contain exponentials and trig functions.

For example, the differential equation $y'' - 4y' + 13y = 0$ has characteristic polynomial $r^2 - 4r + 13$, whose roots are given by the quadratic equation.

$$\begin{aligned} r &= \frac{4 \pm \sqrt{-36}}{2} \\ &= \frac{4 \pm 6i}{2} \\ &= 2 \pm 3i \end{aligned}$$

The root $r = 2 + 3i$ corresponds to the following solution:

$$\begin{aligned} C_{2+3i,1}e^{(2+3i)x} &= C_{2+3i,1}e^{2x}e^{3ix} \\ &= C_{2+3i,1}e^{2x}\left(\cos 3x + i\sin 3x\right) \\ &\to C_{2+3i,1}e^{2x}\left(\cos 3x + \sin 3x\right) \qquad \text{(remove i)} \end{aligned}$$

Likewise, the root $r = 2 - 3i$ corresponds to the following solution:

$$\begin{aligned} C_{2-3i,1}e^{(2-3i)x} &= C_{2-3i,1}e^{2x}e^{-3ix} \\ &= C_{2-3i,1}e^{2x}\left(\cos(-3x) + i\sin(-3x)\right) \\ &= C_{2-3i,1}e^{2x}\left(\cos 3x - i\sin 3x\right) \\ &\to C_{2-3i,1}e^{2x}\left(\cos 3x - \sin 3x\right) \qquad \text{(remove i)} \end{aligned}$$

Assigning new constants $C_1 = C_{2+3i,1} + C_{2-3i,1}$ and $C_2 = C_{2+3i,1} - C_{2-3i,1}$, the full solution becomes the following:

$$\begin{aligned}
&C_{2+3i,1}e^{2x}(\cos 3x + \sin 3x) + C_{2-3i,1}e^{2x}(\cos 3x - \sin 3x) \\
&= e^{2x}\left((C_{2+3i,1} + C_{2-3i,1})\cos 3x + (C_{2+3i,1} - C_{2-3i,1})\sin 3x\right) \\
&= e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)
\end{aligned}$$

Repeated Imaginary Roots

Repeated imaginary and complex roots are treated just like we treated repeated real roots.

For example, the equation $y^{(6)} + 3y^{(4)} + 3y^{(2)} + 1 = 0$ has characteristic polynomial $r^6 + 3r^4 + 3r^2 + 1$, which factors to $(r^2+1)^3$, and thus has roots $r = \pm i$, each with multiplicity 3. The solution to this differential equation is then

$$(C_{i,1} + C_{i,2}x + C_{i,3}x^2)e^{ix} + (C_{-i,1} + C_{-i,2}x + C_{-i,3}x^2)e^{-ix}.$$

After removing the i and grouping the constants, the solution simplifies to

$$(C_1 + C_2x + C_3x^2)\cos x + (C_4 + C_5x + C_6x^2)\sin x.$$

Derivation of the Characteristic Polynomial

Lastly, let's gain a better understanding of why the characteristic polynomial method works. The characteristic polynomial really just comes from guessing a solution $y = Ce^{rx}$. The derivatives for this guess are listed below.

$$\begin{aligned}
y &= Ce^{rx} \\
y' &= Cre^{rx} \\
y'' &= Cr^2e^{rx} \\
&\vdots \\
y^{(n)} &= Cr^ne^{rx}
\end{aligned}$$

We substitute the derivatives in the differential equation, and simplify.

$$\begin{aligned}
0 &= a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y \\
0 &= a_n C r^n e^{rx} + \cdots + a_2 C r^2 e^{rx} + a_1 C r e^{rx} + a_0 e^{rx} \\
0 &= Ce^{rx}\left(a_n r^n + \cdots + a_2 r^2 + a_1 r + a_0\right) \\
0 &= a_n r^n + \cdots + a_2 r^2 + a_1 r + a_0
\end{aligned}$$

We see that $y = Ce^{rx}$ is a solution whenever r is a root of the characteristic polynomial.

Exercises

Use the characteristic polynomial to solve the following differential equations.

1) $y'' + y' - 12y = 0$

2) $2y'' + 16y' + 30y = 0$

3) $y'' + 16y = 0$

4) $y''' - y'' + 2y' - 2y = 0$

5) $y'' - 4y' + 5y = 0$

6) $y'' + 2y' + 17y = 0$

7) $y'' - 4y' + 4y = 0$

8) $y^{(6)} - y^{(4)} = 0$

9) $y^{(5)} + y^{(3)} = 0$

10) $y^{(5)} + y^{(4)} - 2y^{(3)} = 0$

3.5 Undetermined Coefficients

In the previous chapter, we learned how to solve differential equations of the form

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

Now, we consider differential equations of the form

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

where the right hand side is no longer strictly 0, but rather some function $f(x)$. The solution to such a differential equation is given by

$$y(x) = y_0(x) + y_f(x)$$

where y_0 is the general solution to the "homogeneous" equation

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

and y_f is a particular solution that satisfies the "inhomogeneous" equation

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(x).$$

We already know how to use the characteristic polynomial to find y_0, and now we will learn how to use the method of **undetermined coefficients** to find y_f.

The method of undetermined coefficients involves guessing a solution $y_f(x)$ having the same form as $f(x)$, except possibly multiplied by some other coefficients. We then substitute this guess into the differential equation, and solve for the value of the coefficient that will make the guess correct.

Case of Exponential Function

For example, to find a particular solution to the differential equation $y'' + 2y' + y = 1 - 2e^{3x}$, we can guess that $y_f(x) = A + Be^{3x}$ for some values of A and B. Substituting our guess into the equation, we can solve for the correct values of A and B.

$$\begin{aligned} y'' + 2y' + y &= 1 - 2e^{3x} \\ \left(A + Be^{3x}\right)'' + 2\left(A + Be^{3x}\right)' + A + Be^{3x} &= 1 - 2e^{3x} \\ 9Be^{3x} + 6Be^{3x} + A + Be^{3x} &= 1 - 2e^{3x} \\ 16Be^{3x} + A &= 1 - 2e^{3x} \\ A = 1,\ \ B &= -\frac{1}{8} \end{aligned}$$

Our particular solution is then given by $y_f(x) = 1 - \frac{1}{8}e^{3x}$. Then, using the characteristic polynomial method, we solve $y'' + 2y' + y = 0$ to find $y_0(x) = (C_1 + C_2 x)e^{-x}$. The full solution to the differential equation $y'' + 2y' + y = 1 - 2e^{3x}$ is then given by

$$\begin{aligned} y(x) &= y_0(x) + y_f(x) \\ &= (C_1 + C_2 x)e^{-x} + 1 - \frac{1}{8}e^{3x} \end{aligned}.$$

Case of Trig Functions with Same Angle

In cases where $f(x)$ contains $\sin\theta$ or $\cos\theta$, we include both $\sin\theta$ and $\cos\theta$ in our guess for y_0.

For example, to find a particular solution to the differential equation $y''' - 3y'' + 2y' = -3\cos 2x$, we need to construct a guess that contains both $\sin 2x$ and $\cos 2x$. Our guess, then, is

$$y_f(x) = A\sin 2x + B\cos 2x.$$

We substitute this guess into the differential equation and simplify.

$$\begin{aligned} \begin{matrix} (A\sin 2x + B\cos 2x)''' - 3\,(A\sin 2x + B\cos 2x)'' \\ +2\,(A\sin 2x + B\cos 2x)' \end{matrix} &= -3\cos 2x \\ \begin{matrix} (-8A\cos 2x + 8B\sin 2x) - 3\,(-4A\sin 2x - 4B\cos 2x) \\ +2\,(2A\cos 2x - 2B\sin 2x) \end{matrix} &= -3\cos 2x \\ (12A + 4B)\sin 2x + (-4A + 12B)\cos 2x &= -3\cos 2x \end{aligned}$$

Equating coefficients on the left and right sides of the equation yields a system of equations for A and B.

$$\begin{cases} 12A + 4B &= 0 \\ -4A + 12B &= -3 \end{cases}$$

Solving this system, we find $A = \frac{3}{40}$ and $B = -\frac{9}{40}$. The particular solution is then $y_f(x) = \frac{3}{40}\sin 2x - \frac{9}{40}\cos 2x$.

Using the characteristic polynomial to solve $y''' - 3y'' + 2y' = 0$ yields $y_0(x) = C_1 + C_2 e^x + C_3 e^{2x}$, and the full solution of the differential equation $y''' - 3y'' + 2y' = -3\cos 2x$ is then given by

$$\begin{aligned} y(x) &= y_0(x) + y_f(x) \\ &= C_1 + C_2 e^x + C_3 e^{2x} + \frac{3}{40}\sin 2x - \frac{9}{40}\cos 2x \end{aligned}.$$

Case of Trig Functions with Different Angles

When we have multiple values of θ, we end up with even more unknown coefficients in our guess.

For example, to find a particular solution to the differential equation $y'' - 2y = 4\sin 3x - \cos 5x$, we need to construct a guess that contains both $\sin\theta$ and $\cos\theta$, for both $\theta = 3x$ and $\theta = 5x$. Our guess, then, is

$$y_f(x) = A\sin 3x + B\cos 3x + C\sin 5x + D\cos 5x.$$

We substitute this guess into the differential equation and simplify.

$$\begin{aligned} \begin{matrix} (A\sin 3x + B\cos 3x + C\sin 5x + D\cos 5x)'' \\ -2\left(A\sin 3x + B\cos 3x + C\sin 5x + D\cos 5x\right) \end{matrix} &= 4\sin 3x - \cos 5x \\ \begin{matrix} (-9A\sin 3x - 9B\cos 3x - 25C\sin 5x - 25D\cos 5x) \\ -2\left(A\sin 3x + B\cos 3x + C\sin 5x + D\cos 5x\right) \end{matrix} &= 4\sin 3x - \cos 5x \\ -11A\sin 3x - 11B\cos 3x - 27C\sin 5x - 27D\cos 5x &= 4\sin 3x - \cos 5x \end{aligned}$$

Equating coefficients on the left and right sides of the equation yields $A = -\frac{4}{11}$, $B = 0$, $C = 0$, and $D = \frac{1}{27}$. The particular solution is then $y_f(x) = -\frac{4}{11}\sin 3x + \frac{1}{27}\cos 5x$.

Using the characteristic polynomial to solve $y'' - 2y = 0$ yields $y_0(x) = C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x}$, and the full solution of the differential equation $y'' - 2y = 4\sin 3x - \cos 5x$ is then given by

$$\begin{aligned} y(x) &= y_0(x) + y_f(x) \\ &= C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x} - \frac{4}{11}\sin 3x + \frac{1}{27}\cos 5x \end{aligned}.$$

Case of Polynomial Functions

Lastly, the differential equation $y'' + y' = x^3 - 2x + e^{2x}$ has a polynomial and an exponential term, so our guess for the particular solution needs to contain a polynomial and an exponential term.

The polynomial in the differential equation is of degree 3, and the differential equation itself is of degree 2, so our guess needs to contain a polynomial of degree $3+2=5$.

$$y_f(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F + Ge^{2x}$$

We substitute this guess into the differential equation and simplify.

$$\begin{aligned}
\begin{matrix} \left(Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F + Ge^{2x}\right)'' \\ + \left(Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F + Ge^{2x}\right)' \end{matrix} &= x^3 - 2x + e^{2x} \\
\begin{matrix} \left(20Ax^3 + 12Bx^2 + 6Cx + 2D + 4Ge^{2x}\right) \\ + \left(5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E + 2Ge^{2x}\right) \end{matrix} &= x^3 - 2x + e^{2x} \\
\begin{matrix} 5Ax^4 + (20A + 4B)x^3 + (12B + 3C)x^2 \\ +(6C + 2D)x + 2D + E + 6Ge^{2x} \end{matrix} &= x^3 - 2x + e^{2x}
\end{aligned}$$

Equating coefficients on the left and right sides of the equation yields $A=0$, $B=\frac{1}{4}$, $C=-1$, $D=2$, $E=-4$, and $G=\frac{1}{6}$. The coefficient F can still be any number, so we leave it as-is. The particular solution is then

$$y_f(x) = \frac{1}{4}x^4 - x^3 + 2x^2 - 4x + F + \frac{1}{6}e^{2x}.$$

Using the characteristic polynomial to solve $y'' + y' = 0$ yields $y_0(x) = C_1 + C_2e^{-x}$, and the full solution of the differential equation $y'' + y' = x^3 - 2x + e^{2x}$ is then given by

$$\begin{aligned}
y(x) &= y_0(x) + y_f(x) \\
&= C_1 + C_2e^{-x} + \frac{1}{4}x^4 - x^3 + 2x^2 - 4x + F + \frac{1}{6}e^{2x}.
\end{aligned}$$

To eliminate redundancy, we can lump the F constant into the C_1 constant, since $C_1 + F$ is itself just another constant.

$$y(x) = C_1 + C_2 e^{-x} + \frac{1}{4}x^4 - x^3 + 2x^2 - 4x + \frac{1}{6}e^{2x}$$

Exercises

Use the method of undetermined coefficients to solve the following differential equations.

1) $y'' + y = 4e^{5x}$

2) $y' + 3y = \sin(2x)$

3) $y'' - y' = \cos(\pi x)$

4) $y''' - 2y' = x^2 + 1$

5) $2y' - y = \sin(x) - \cos(2x)$

6) $2y' + y = e^x + 3\sin(x)$

7) $4y'' - 9y = 2x^4 - 3x^2 + \cos(x+1)$

8) $y' + y = \sin(2x+1) + \cos(5x) + 1$

3.6 Integrating Factors

We know how to solve differential equations of the form

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

where each coefficient a_i is a constant. In this chapter, we consider differential equations of the form

$$y' + a(x)y = f(x)$$

where the coefficient $a(x)$ is itself a function of x.

To solve such equations using the method of **integrating factors**, we start off multiplying both sides of the equation by the term $e^{\int a(x)\,dx}$, which is known as the **integrating factor**. Then, we can write the left hand side as the derivative of $ye^{\int a(x)\,dx}$, antidifferentiate, and solve for y.

$$\begin{aligned}
y' + a(x)y &= f(x) \\
y' e^{\int a(x)\,dx} + a(x) y e^{\int a(x)\,dx} &= f(x) e^{\int a(x)\,dx} \\
\left(y e^{\int a(x)\,dx} \right)' &= f(x) e^{\int a(x)\,dx} \\
y e^{\int a(x)\,dx} &= \int f(x) e^{\int a(x)\,dx}\,dx \\
y &= e^{-\int a(x)\,dx} \int f(x) e^{\int a(x)\,dx}\,dx
\end{aligned}$$

Demonstration

For example, consider the differential equation $y' - \frac{3}{x}y = 2x + 1$. The integrating factor for this equation is as follows:

$$\begin{aligned} e^{\int -\frac{3}{x}\,dx} &= e^{-3\ln x} \\ &= \left(e^{\ln x}\right)^{-3} \\ &= x^{-3} \\ &= \frac{1}{x^3} \end{aligned}$$

To solve the equation, we multiply both sides of the equation, group the derivative, take the antiderivative, and solve for y.

$$\begin{aligned} y' - \frac{3}{x}y &= 2x + 1 \\ \frac{1}{x^3}y' - \frac{1}{x^3}\cdot\frac{3}{x}y &= \frac{1}{x^3}\cdot(2x+1) \\ \frac{1}{x^3}y' - \frac{3}{x^4}y &= \frac{2}{x^2} + \frac{1}{x^3} \\ \left(\frac{1}{x^3}y\right)' &= \frac{2}{x^2} + \frac{1}{x^3} \\ \frac{1}{x^3}y &= \int \frac{2}{x^2} + \frac{1}{x^3}\,dx \\ \frac{1}{x^3}y &= -\frac{2}{x} - \frac{1}{2x^2} + C \\ y &= -2x^2 - \frac{1}{2}x + Cx^3 \end{aligned}$$

Case when Leading Coefficient is Not One

In equations where the coefficient on the y' is not already 1, we need to start by dividing the equation by that coefficient.

For example, to solve the equation $xy' + \frac{y}{\ln x} = x^3$, we start by dividing by x, which yields $y' + \frac{y}{x \ln x} = x^2$. Then, we can proceed as usual to calculate the integration factor.

$$\begin{aligned} e^{\int \frac{1}{x \ln x}\, dx} &= e^{\ln(\ln x)} \\ &= \ln x \end{aligned}$$

Now, we can multiply our updated equation by the integration factor, and solve for y (using integration by parts along the way).

$$\begin{aligned} y' + \frac{y}{x \ln x} &= x^2 \\ y' \ln x + \frac{y}{x \ln x} \cdot \ln x &= x^2 \ln x \\ y' \ln x + \frac{y}{x} &= x^2 \ln x \\ (y \ln x)' &= x^2 \ln x \\ y \ln x &= \int x^2 \ln x \, dx \\ y \ln x &= \frac{1}{3} x^3 \ln x - \int \frac{x^2}{3}\, dx \\ y \ln x &= \frac{1}{3} x^3 \ln x - \frac{x^3}{9} + C \\ y &= \frac{x^3}{3} - \frac{x^3}{9 \ln x} + \frac{C}{\ln x} \end{aligned}$$

Exercises

Use integrating factors to solve the following differential equations.

1) $y' + \dfrac{y}{x} = \sin x$

2) $y' + \dfrac{y}{x \ln x} = \ln x$

3) $y' + y \cot x = 1$

4) $xy' + y = \sec^2(x)$

5) $xy' + y = \sec^2(x)$

6) $y' \tan x - y = \sec x$

3.7 Variation of Parameters

When we know the zero solutions y_0 of a differential equation $y'' + a_1(x)y' + a_0(x)y = f(x)$, we can use a method called **variation of parameters** to find the particular solution. This method is especially useful in cases where we are unable to guess the particular solution through undetermined coefficients.

Derivation

Variation of parameters is similar to undetermined coefficients in that we guess a solution form that is relevant to the differential equation, and adjust it as needed to solve the differential equation.

However, variation of parameters is more general: the guess is of the form $y_f(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where y_1 and y_2 are the two zero solutions of the differential equation $y'' + a_1(x)y' + a_0(x)y = 0$, and $u_1(x)$ and $u_2(x)$ are some unknown multiplier functions for which we need to solve.

If we also force $y_f'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x)$, then we can set up a system of equations to solve for u_1 and u_2. (To be clear, the formula for y_f' does not come from differentiating -- rather, it is a condition that we force, so that we obtain a solvable system of equations.)

The first equation comes from differentiating y_f:

$$\begin{aligned} y_f' &= u_1 y_1' + u_2 y_2' \\ (u_1 y_1 + u_2 y_2)' &= u_1 y_1' + u_2 y_2' \\ u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' &= u_1 y_1' + u_2 y_2' \\ u_1' y_1 + u_2' y_2 &= 0 \end{aligned}$$

The second equation comes from substituting our guess for y_f into the differential equation and simplifying, using the fact that y_1 and y_2 are the zero solutions.

$$\begin{aligned} &f = y_f'' + a_1 y_f' + a_0 y_f \\ &f = (u_1 y_1' + u_2 y_2')' + a_1(u_1 y_1' + u_2 y_2') + a_0(u_1 y_1 + u_2 y_2) \\ &f = (u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'') + a_1(u_1 y_1' + u_2 y_2') + a_0(u_1 y_1 + u_2 y_2) \\ &f = (u_1)(y_1'' + a_1 y_1' + a_0 y_1) + (u_2)(y_2'' + a_1 y_2' + a_0 y_2) + u_1' y_1' + u_2' y_2' \\ &f = (u_1)(0) + (u_2)(0) + u_1' y_1' + u_2' y_2' \\ &f = u_1' y_1' + u_2' y_2' \end{aligned}$$

Our resulting system

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f \end{cases}$$

is solved by

$$\begin{aligned} u_1' &= -\frac{y_2 f}{y_1 y_2' - y_2 y_1'} \\ u_2' &= \frac{y_1 f}{y_1 y_2' - y_2 y_1'} \end{aligned} .$$

Integrating, we have

$$\begin{aligned} u_1 &= -\int \frac{y_2 f}{y_1 y_2' - y_2 y_1'}\, dx \\ u_2 &= \int \frac{y_1 f}{y_1 y_2' - y_2 y_1'}\, dx \end{aligned}.$$

The particular solution is then

$$\begin{aligned} y_f &= u_1 y_1 + u_2 y_y \\ &= -y_1 \int \frac{y_2 f}{y_1 y_2' - y_2 y_1'}\, dx + y_2 \int \frac{y_1 f}{y_1 y_2' - y_2 y_1'}\, dx \end{aligned}.$$

Demonstration

For example, to solve the differential equation $y'' - 2y' + y = \frac{e^x}{x}$, we start by solving $y'' - 2y' + y = 0$ to find the zero solutions $y_1 = e^x$ and $y_2 = xe^x$. After computing

$$\begin{aligned} y_1 y_2' - y_2 y_1' &= e^x(e^x + xe^x) - xe^x(e^x) \\ &= e^{2x} \end{aligned}$$

we are able to compute u_1 and u_2:

$$\begin{aligned}
u_1 &= -\int \frac{y_2 f}{y_1 y_2' - y_2 y_1'}\,dx \\
&= -\int \frac{xe^x\left(\frac{e^x}{x}\right)}{e^{2x}}\,dx \\
&= -\int 1\,dx \\
&= -x
\end{aligned}$$

$$\begin{aligned}
u_2 &= \int \frac{y_1 f}{y_1 y_2' - y_2 y_1'}\,dx \\
&= \int \frac{e^x\left(\frac{e^x}{x}\right)}{e^{2x}}\,dx \\
&= \int \frac{1}{x}\,dx \\
&= \ln x
\end{aligned}$$

We can then compute the particular solution:

$$\begin{aligned}
y_f &= u_1 y_1 + u_2 y_2 \\
&= (-x)(e^x) + (\ln x)(xe^x) \\
&= -xe^x + xe^x \ln x \\
&= xe^x(\ln x - 1)
\end{aligned}$$

Finally, we can write the full solution, and lump any constant terms to eliminate redundancy.

$$\begin{aligned}
y &= y_0 + y_f \\
&= (C_1 + C_2 x)e^x + e^x(x\ln x - 1) \\
&= (C_1 + C_2 x + x\ln x)e^x
\end{aligned}$$

Another Demonstration

As another example, we solve the differential equation $y'' - y' = xe^x \sin x$ in the same way. The zero solutions to $y'' - y' = 0$ are $y_1 = e^x$ and $y_2 = 1$, and we have

$$\begin{aligned} y_1 y_2' - y_2 y_1' &= (e^x)(0) - (1)(e^x) \\ &= -e^x \end{aligned}.$$

Computing u_1 and u_2, we have

$$\begin{aligned} u_1 &= -\int \frac{y_2 f}{y_1 y_2' - y_2 y_1'}\, dx \\ &= -\int \frac{xe^x \sin x}{-e^x}\, dx \\ &= \int x \sin x \, dx \\ &= \sin x - x \cos x \end{aligned}$$

$$\begin{aligned} u_2 &= \int \frac{y_1 f}{y_1 y_2' - y_2 y_1'}\, dx \\ &= \int \frac{e^x \cdot xe^x \sin x}{-e^x}\, dx \\ &= -\int xe^x \sin x \, dx \\ &= \frac{1}{2} e^x \left(x \cos x - x \sin x - \cos x \right) \end{aligned}.$$

We can then compute the particular solution:

$$\begin{aligned}
y_f &= u_1 y_1 + u_2 y_2 \\
&= (\sin x - x\cos x)e^x + \frac{1}{2}e^x(x\cos x - x\sin x - \cos x) \\
&= \frac{1}{2}e^x\left(2\sin x - x\cos x - x\sin x - \cos x\right)
\end{aligned}$$

Finally, we can write the full solution, and lump any constant terms to eliminate redundancy.

$$\begin{aligned}
y &= y_0 + y_f \\
&= C_1 e^x + C_2 + \frac{1}{2}e^x\left(2\sin x - x\cos x - x\sin x - \cos x\right) \\
&= C_2 + \frac{1}{2}e^x\left(C_1 + 2\sin x - x\cos x - x\sin x - \cos x\right)
\end{aligned}$$

Exercises

Use variation of parameters to solve the following differential equations.

1) $y'' - 2y' + y = \dfrac{e^x}{x^2}$

2) $y'' - 2y' + y = e^x \ln x$

3) $y'' - 4y = \dfrac{1}{1 + e^{2x}}$

4) $y'' + 2y' + y = \dfrac{1}{(1 + x^2)e^x}$

5) $y'' + y = xe^x \cos x$

6) $y'' - y' = x^2 e^x \sin x$

Part 4
Series

4.1 Geometric Series

A **geometric series** is a sum of the form $r + r^2 + r^3 + \cdots$ for some number r.

Convergence

For example, when $r = \frac{1}{2}$, the corresponding geometric series is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. This series might look like it grows bigger and bigger as you add more terms, but there is actually a limit to how big it can get.

To understand the limit intuitively, think of each term as representing a section of a pie. First, you eat half of the pie, $\frac{1}{2}$. Next, you eat half of the remaining half, $\frac{1}{4}$. Then, you eat half of the remaining quarter, $\frac{1}{8}$, and so on, eating half of what's left every time.

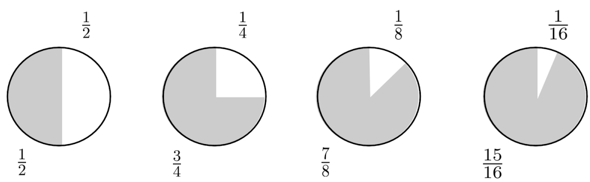
You'll never finish the pie, because there will always be something left over -- but in the limit as the number of terms approaches infinity, the leftover piece shrinks to 0, and the amount of pie that you consume approaches 1. This means that the sum of the terms is 1, and we say that the series **converges** to 1.

$$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$



Divergence

On the other hand, the series for $r=2$ legitimately blows up to infinity -- the terms keep getting bigger and bigger, so the sum has to keep getting bigger and bigger. We say that the series **diverges** to infinity.

$$2+4=6$$

$$2+4+8=14$$

$$2+4+8+16=30$$

$$2+4+8+16+\cdots=\infty$$

Computing the Sum

But what about the series for, say, $r=0.9$? It's not so obvious whether it converges or diverges. Even if we're told that it converges, what number does it converge to? We can compute this algebraically.

$$\begin{aligned}
r &= r+r^2-r^2+r^3-r^3+\cdots \\
r &= (r+r^2+r^3+\cdots)-(r^2+r^3+r^4+\cdots) \\
r &= (r+r^2+r^3+\cdots)-r(r+r^2+r^3+\cdots) \\
r &= (1-r)(r+r^2+r^3+\cdots) \\
\frac{r}{1-r} &= r+r^2+r^3+\cdots
\end{aligned}$$

We can check our formula by making sure it evaluates to 1 when given $r=\frac{1}{2}$.

$$\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$$

We can also use our the formula to find what the series with $r = 0.9$ converges to.

$$0.9 + 0.9^2 + 0.9^3 + \cdots = \frac{0.9}{1 - 0.9} = 9.$$

Understanding Nonsensical Results

But there's one issue -- the formula gives a finite result for $r = 2$, which we know diverges to infinity since each additional term is bigger than the previous term. According to the formula, the series with $r = 2$ should converge to -2, which doesn't make any sense.

$$2 + 2^2 + 2^3 + \cdots = \frac{2}{1 - 2} = -2 \qquad \text{(invalid)}$$

In general, the formula only gives the correct result if the series converges, and the series only converges when $|r| < 1$. (We'll see why in a moment.)

When the series diverges, we can get nonsense results from the formula because the method by which the formula was obtained is no longer valid. Algebra doesn't work on terms that diverge to infinity -- for example, it's true that $\infty + 1 = \infty$, but subtracting ∞ from both sides of the equation leads to the statement $1 = 0$, which isn't true.

Determining Convergence

To see why the geometric series only converges when $|r| < 1$, we need to compute the sum formula again, but this time only for the first n terms of the series, so that we don't run into any problems with divergence.

$$\begin{aligned}
r - r^{n+1} &= r - r^2 + r^2 - r^3 + r^3 - \cdots - r^n + r^n - r^{n+1} \\
r - r^{n+1} &= (r + r^2 + r^3 + \cdots + r^n) - (r^2 + r^3 + r^4 + \cdots + r^{n+1}) \\
r - r^{n+1} &= (r + r^2 + r^3 + \cdots + r^n) - r(r + r^2 + r^3 + \cdots + r^n) \\
r - r^{n+1} &= (1 - r)(r + r^2 + r^3 + \cdots + r^n) \\
\frac{r - r^{n+1}}{1 - r} &= r + r^2 + r^3 + \cdots
\end{aligned}$$

Now, we can find the sum of the full series by taking the limit as $n \to \infty$.

$$\begin{aligned}
r + r^2 + r^3 + \cdots &= \lim_{n\to\infty} \frac{r - r^{n+1}}{1 - r} \\
&= \frac{r - \lim\limits_{n\to\infty} r^{n+1}}{1 - r}
\end{aligned}$$

In order for $\lim\limits_{n\to\infty} r^{n+1}$ to converge and the denominator $1 - r$ not to go to 0, we require that $|r| < 1$.

Case when Manipulation is Required

Lastly, sometimes we may have to factor out and/or separate numbers from a geometric series in order to find its sum.

For example, to find the sum of the geometric series

$$2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\ldots$$

we can factor out a 2 and separate the first term from the rest of the series. Then, we can apply the sum formula to the rest of the series and simplify the expression.

$$\begin{aligned}
&2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\ldots \\
&=2\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\right) \\
&=2\left(1+\frac{\frac{1}{3}}{1-\frac{1}{3}}\right) \\
&=2\left(1+\frac{\frac{1}{3}}{\frac{2}{3}}\right) \\
&=2\left(1+\frac{1}{2}\right) \\
&=3
\end{aligned}$$

Exercises

Compute the sum of each series.

1) $\dfrac{1}{3}+\dfrac{1}{9}+\dfrac{1}{27}+\ldots$

2) $1+2+3+\ldots$

3) $\dfrac{1}{5}+\dfrac{1}{25}+\dfrac{1}{125}+\ldots$

4) $\dfrac{2}{3}+\dfrac{4}{9}+\dfrac{8}{27}+\ldots$

5) $\dfrac{3}{2}+\dfrac{9}{4}+\dfrac{27}{8}+\ldots$

6) $0.01+0.01+0.01+\ldots$

7) $0.9+0.09+0.009+\ldots$

8) $0.9+0.009+0.00009+\ldots$

4.2 Tests for Convergence

Previously, we saw that sum formulas are only valid for series that converge. But how can we tell whether a series converges or diverges, in the first place?

Trivial Test

First of all, an easy way to tell that a series diverges is to look at the terms of the series -- if the terms themselves do not converge to 0, then their sum cannot possibly converge.

But if the terms do converge to 0, then we can't tell whether the series converges or diverges, and we have to use a more powerful test.

Integral Test

The **integral test** is a powerful test for proving convergence. It says that if the series can be written as $f(1)+f(2)+f(3)+\ldots$ for some decreasing function f, then the series converges if the integral $\int_1^\infty f(x)\,dx$ converges, and diverges if the integral $\int_1^\infty f(x)\,dx$ diverges.

For example, to tell whether the series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots$$

conveges, we can perform the integral test with $\int_1^\infty \frac{1}{x}\,dx$. This integral diverges to infinity, so the series above diverges to infinity as well.

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= [\ln x]_1^\infty \\ &= \ln\infty - \ln 1 \\ &= \infty - 0 \\ &= \infty\end{aligned}$$

On the other hand, applying the integral test to the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

shows that the series converges. (But the series does not converge to the same value of the integral -- the integral test can tell us that a series converges, but not the value to which it converges. In general, the value to which a series converges may be difficult to compute.)

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \left[-\frac{1}{x}\right]_1^\infty \\ &= -\frac{1}{\infty}-\left(-\frac{1}{1}\right) \\ &= 0-(-1) \\ &= 1\end{aligned}$$

More generally, considering all exponents in the denominator, we can use the integral test to show that any series of the form

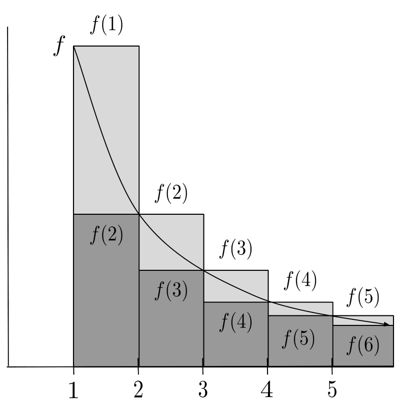
$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

converges when $p > 1$ (and diverges otherwise).

Derivation of the Integral Test

The integral test works because the value of the integral is bounded above by the series, and below by the series excluding the first term.

$$f(2) + f(3) + f(4) + \dots < \int_1^\infty f(x)\,dx < f(1) + f(2) + f(3) + \dots$$



If the integral converges, then the series excluding the first term must converge, and adding a single finite term to the series cannot affect convergence, so the series in full must converge.

On the other hand, if the integral diverges, then since the series is greater than the integral, the series must also diverge.

Ratio Test

Another powerful test for proving convergence is the **ratio test**, which does not require any integration and thus can handle hard-to-integrate series.

The ratio test says that if the ratio of terms in a series has a limit r, then the series is almost like a geometric series with ratio r -- it converges if $|r| < 1$, and diverges if $|r| > 1$. The only catch is that if $r = 1$, then we can't tell whether the series converges or diverges (whereas a geometric series with $r = 1$ must diverge).

For example, consider the following series:

$$3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \ldots$$

The n$^{\text{th}}$ term of this series is given by $\frac{3^n}{n!}$, and the ratio of the terms has a limit of 0, so the ratio test tells us that the series converges.

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} &= \lim_{n\to\infty}\frac{n!\cdot 3^{n+1}}{(n+1)!\cdot 3^n}\\ &= \lim_{n\to\infty}\frac{3}{n+1}\\ &= 0\end{aligned}$$

On the other hand, the n^th term of the series

$$-\frac{1}{10}+\frac{2!}{10^2}-\frac{3!}{10^3}+\frac{4!}{10^4}-\ldots$$

is given by $\frac{n!\cdot(-1)^n}{10^n}$, and the ratio of the terms has a limit that diverges to $-\infty$, so the ratio test tells us that the series diverges.

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{(n+1)!\cdot(-1)^{n+1}}{10^{n+1}}}{\frac{n!\cdot(-1)^n}{10^n}} &= \lim_{n\to\infty}\frac{(n+1)!\cdot(-1)^{n+1}\cdot 10^n}{n!\cdot(-1)^n\cdot 10^{n+1}}\\ &= \lim_{n\to\infty}\frac{(n+1)\cdot(-1)}{10}\\ &= \lim_{n\to\infty}-\frac{n+1}{10}\\ &= -\infty\end{aligned}$$

Root Test

Yet another test for convergence, called the **root test**, says that if the n^th root of the n^th term of the series has a limit r, then it is (once again) almost like a geometric series with ratio r -- it converges if

$|r| < 1$, and diverges if $|r| > 1$. The only catch (once again) is that if $r = 1$, then we can't tell whether the series converges or diverges.

For example, consider the following series:

$$1 + \left(\frac{2}{3}\right)^4 + \left(\frac{3}{5}\right)^6 + \left(\frac{4}{7}\right)^8 + \dots$$

The nth term of this series is given by $\left(\frac{n}{2n-1}\right)^{2n}$, and the nth root of the nth term has a limit of $\frac{1}{4}$, so the root test tells us that the series converges.

$$\begin{aligned}
\lim_{n\to\infty} \left[\left(\frac{n}{2n-1}\right)^{2n}\right]^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\frac{n}{2n-1}\right)^2 \\
&= \left(\lim_{n\to\infty} \frac{n}{2n-1}\right)^2 \\
&= \left(\frac{1}{2}\right)^2 \\
&= \frac{1}{4}
\end{aligned}$$

On the other hand, the nth term of the series

$$\frac{1}{5^3} + \frac{2^2}{5^6} + \frac{3^3}{5^9} + \frac{4^4}{5^{12}} + \dots$$

is given by $\frac{n^n}{5^{3n}}$, and the nth root of the nth term has a limit that diverges to infinity, so the root test tells us that the series diverges.

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{n^n}{5^{3n}}\right]^{\frac{1}{n}} &= \lim_{n\to\infty}\frac{n}{5^3} \\ &= \infty\end{aligned}$$

Limit Comparison Test

Lastly, the **limit comparison** test tells us that for any series $a_1 + a_2 + a_3 + \ldots$, if we create another series $b_1 + b_2 + b_3 + \ldots$ such that $\lim_{n\to\infty}\frac{a_n}{b_n} = c$ for some positive constant c, then either both series converge or both series diverge.

The limit comparison test can simplify the process of finding convergence for complicated series -- for example, given a series with terms $\frac{n+\sin n}{n^2}$, we can construct a new series with terms $\frac{1}{n}$ whose ratio with the original series has a limit of 1.

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{n+\sin n}{n^2}}{\frac{1}{n}} &= \lim_{n\to\infty}\frac{n+\sin n}{n} \\ &= \lim_{n\to\infty}\left[1+\frac{\sin n}{n}\right] \\ &= 1+0 \\ &= 1\end{aligned}$$

Since the series with terms $\frac{1}{n}$ diverges, the original series with terms $\frac{n+\sin n}{n^2}$ must diverge as well.

Likewise, the series with terms $\frac{3n+2}{\sqrt{n^6+1}}$ can be compared to the series with terms $\frac{1}{n^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{3n+2}{\sqrt{n^6+1}}}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{3n^3+2n^2}{\sqrt{n^6+1}} \\
&= \lim_{n\to\infty} \sqrt{\frac{(3n^3+2n^2)^2}{n^6+1}} \\
&= \lim_{n\to\infty} \sqrt{\frac{9n^6+12n^5+4n^4}{n^6+1}} \\
&= \sqrt{\lim_{n\to\infty} \frac{9n^6+12n^5+4n^4}{n^6+1}} \\
&= \sqrt{9} \\
&= 3
\end{aligned}$$

We know the series with terms $\frac{1}{n^2}$ converges, so the original series with terms $\frac{3n+2}{\sqrt{n^6+1}}$ must converge as well.

Exercises

Tell whether each series converges or diverges.

1) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^2-4}$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(3n-2)^2}$

3) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n^2)}{n^2}$

4) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

5) $\displaystyle\sum_{n=1}^{\infty} (n!)e^{-n}$

6) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$

7) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

8) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

9) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^n}$

10) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^{\frac{n}{2}}$

11) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1-9n}{1-2n}\right)^{n}$

12) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}+\frac{1}{n}\right)^{n}$

13) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-n}{n^3+1}$

14) $\displaystyle\sum_{n=1}^{\infty} \frac{n+8}{\sqrt{n^5+2}}$

15) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3-n+2}{(n+4)^2}$

16) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{ne^n}$

4.3 Taylor Series

The sum formula for a geometric series is an example representing a non-polynomial function as an infinite polynomial within a particular range of inputs.

$$\frac{x}{1-x} = x + x^2 + x^3 + \cdots \qquad \text{(provided } |x| < 1\text{)}$$

Many other non-polynomial functions can be represented by infinite polynomials called **Taylor series**. The general formula for the Taylor series of a function $f(x)$, centered about a point $x = c$, is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \, .$$

Just like for the geometric series sum formula, the Taylor series can only be used when it converges. The ratio test is particularly useful for finding the x-values for which the series converges.

For the sake of example, we will compute the Taylor series of several familiar functions: e^x, $\sin x$, and $\ln x$. To introduce some variety, we will center each series at a different x-value.

Taylor Series of the Exponential Function

For $f(x)=e^x$, we have $f'(x)=e^x$, $f''(x)=e^x$, and in general $f^{(n)}(x)=e^x$ for all values of n. The Taylor series of $f(x)=e^x$ centered at $x=0$ is then given by

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= \sum_{n=0}^{\infty} \frac{e^0}{n!}x^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned} .$$

Applying the ratio test, we see that the series converges when

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| &< 1 \\ \lim_{n\to\infty}\left|\frac{x}{n+1}\right| &< 1 \\ 0 &< 1 \end{aligned} .$$

Thus, the series converges for all values of x.

Taylor Series of Sine

For $f(x) = \sin x$, we have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and in general $f^{(2n)}(x) = (-1)^n \sin x$ and $f^{(2n+1)}(x) = (-1)^n \cos x$.

The Taylor series of $f(x) = \sin x$ centered at $x = \pi$ is then given by

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n \\
&= \sum_{n=0}^{\infty} \frac{f^{(2n)}(\pi)}{(2n)!}(x-\pi)^{2n} + \frac{f^{(2n+1)}(\pi)}{(2n+1)!}(x-\pi)^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \sin(\pi)}{(2n)!}(x-\pi)^{2n} + \frac{(-1)^n \cos(\pi)}{(2n+1)!}(x-\pi)^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (0)}{(2n)!}(x-\pi)^{2n} + \frac{(-1)^n (-1)}{(2n+1)!}(x-\pi)^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}
\end{aligned}
.
$$

Applying the ratio test, we see that the series converges when

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)+1}}{(2(n+1)+1)!}(x-\pi)^{2(n+1)+1}}{\frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}}\right| &< 1 \\
\lim_{n\to\infty}\left|\frac{\frac{(x-\pi)^{2n+3}}{(2n+3)!}}{\frac{(x-\pi)^{2n+1}}{(2n+1)!}}\right| &< 1 \\
\lim_{n\to\infty}\left|\frac{(x-\pi)^2}{(2n+2)(2n+3)}\right| &< 1 \\
0 &< 1.
\end{aligned}$$

Thus, the series converges for all values of x.

Taylor Series of Natural Log

For $f(x) = \ln x$, we have $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, and in general $f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$ for $n \geq 1$.

The Taylor series of $f(x) = \ln x$ centered at $x = 1$ is then given by

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= f(1) + \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= \ln 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\frac{(n-1)!}{1^n}}{n!}(x-1)^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \end{aligned}$$.

Applying the ratio test, we see that the series converges when

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{(-1)^{(n+1)-1}}{n+1}(x-1)^{n+1}}{\frac{(-1)^{n-1}}{n}(x-1)^n} \right| &< 1 \\ \lim_{n\to\infty} \left| \frac{n}{n+1}(x-1) \right| &< 1 \\ |x-1| \lim_{n\to\infty} \left| \frac{n}{n+1} \right| &< 1 \\ |x-1| &< 1 \end{aligned}$$.

Thus, the series converges for $0 < x < 2$.

Derivation

To see where the formula for the Taylor series comes from, we start by performing repeated integration on the function $f^{(n+1)}(x)$.

$$\begin{aligned}
&\underbrace{\int_c^x \cdots \int_c^x}_{\text{n+1 integrals}} f^{(n+1)}(x)\,(dx)^{n+1} \\
&= \underbrace{\int_c^x \cdots \int_c^x}_{\text{n integrals}} f^{(n)}(x) - f^{(n)}(c)\,(dx)^n \\
&= \underbrace{\int_c^x \cdots \int_c^x}_{\text{n-1 integrals}} f^{(n-1)}(x) - f^{(n-1)}(c) - f^{(n)}(c)(x-c)\,(dx)^{n-1} \\
&= \underbrace{\int_c^x \cdots \int_c^x}_{\text{n-2 integrals}} f^{(n-2)}(x) - f^{(n-2)}(c) - f^{(n-1)}(c)(x-c) - \frac{f^{(n)}(c)}{2}(x-c)^2\,(dx)^{n-2} \\
&= \quad \cdots \\
&= f(x) - f(c) - f'(c)(x-c) - \frac{f''(c)}{2}(x-c)^2 - \cdots - \frac{f^{(n)}(c)}{n!}(x-c)^n
\end{aligned}$$

Solving for $f(x)$, we find

$$\begin{aligned}
f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\
+ \underbrace{\int_c^x \cdots \int_c^x}_{\text{n+1 integrals}} f^{(n+1)}(x)\,(dx)^{n+1}
\end{aligned}$$
.

Taking the limit as $n \to \infty$, we can express f as the sum of its Taylor series and some remainder term.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n + \lim_{n\to\infty} \underbrace{\int_c^x \cdots \int_c^x}_{\text{n+1 integrals}} f^{(n+1)}(x)\,(dx)^{n+1}$$

For many familiar functions, with x sufficiently close to c, it is often the case that the remainder decays to zero:

$$\lim_{n\to\infty} \underbrace{\int_c^x \cdots \int_c^x}_{\text{n+1 integrals}} f^{(n+1)}(x)\,(dx)^{n+1} = 0$$

For example, the remainder decays to zero if f is any polynomial, because differentiating an n$^{\text{th}}$ degree polynomial $n+1$ times always yields a result of 0, and the integral of 0 is always 0. (But this is rather trivial since the Taylor series of a polynomial is the polynomial itself.)

More generally, we can place an upper bound on the size of the n$^{\text{th}}$ remainder:

$$\left| \underbrace{\int_c^x \cdots \int_c^x}_{\text{n+1 integrals}} f^{(n+1)}(x)\,(dx)^{n+1} \right| \leq \left(\max_{[c,x]} \left| f^{(n+1)} \right| \right) \left| \underbrace{\int_c^x \cdots \int_c^x}_{\text{n+1 integrals}} 1\,(dx)^{n+1} \right|$$

Then, since

$$\underbrace{\int_c^x \cdots \int_c^x}_{\text{n+1 integrals}} 1\,(dx)^{n+1} = \frac{(x-c)^{n+1}}{(n+1)!}$$

we must have that

$$\left| \lim_{n\to\infty} \underbrace{\int_c^x \cdots \int_c^x}_{\text{n+1 integrals}} f^{(n+1)}(x)\,(dx)^{n+1} \right| \le \lim_{n\to\infty} \left(\max_{[c,x]} \left| f^{(n+1)} \right| \right) \frac{|x-c|^{n+1}}{(n+1)!}$$

.

Provided that the (n+1)st derivative doesn't grow large enough to overpower the $(n+1)!$ term in the denominator as $n \to \infty$, the remainder will decay to zero. Then the function will be equal to its Taylor series, provided that the series converges.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Exercises

Compute the Taylor series for the following functions, centered at the given points. Also compute the interval of convergence.

1) $f(x) = \ln(1+x)$ at $x = 0$

2) $f(x) = \dfrac{1}{(1+x)^2}$ at $x = 0$

3) $f(x) = \cos x$ at $x = 0$

4) $f(x) = \cos x$ at $x = \pi$

5) $f(x) = \arctan x$ at $x = 0$

6) $f(x) = 2^x$ at $x = -1$

7) $f(x) = \dfrac{1}{x}$ at $x = 2$

8) $f(x) = \dfrac{1}{x}$ at $x = -10$

4.4 Manipulating Taylor Series

To find the Taylor series of complicated functions, it's often easiest to manipulate the Taylor series of simpler functions, such as those given below.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (-1 < x < 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty < x < \infty)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad (-1 < x \leq 1)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad (-\infty < x < \infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad (-\infty < x < \infty)$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad (-1 \leq x \leq 1)$$

Multiplying by a Constant

For example, to compute the Taylor series of xe^x centered at $x=0$, we can take the elementary Taylor series $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and multiply it by x.

$$\begin{aligned} xe^x &= x\sum_{n=0}^{\infty}\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} \end{aligned}$$

Though not strictly necessary, we can make the exponent on the x match the index of summation by changing the index of summation to $k=n+1$.

$$xe^x=\sum_{k=1}^{\infty}\frac{x^k}{(k-1)!}$$

In this case, since we are multiplying the series by a constant, the interval of convergence of the series will stay the same: $-\infty<x<\infty$.

This is because a convergent series has a finite sum, and multiplying by a constant cannot cause a finite number to become infinite; whereas a divergent series has an infinite sum, and multiplying by a constant cannot cause an infinite number to become finite.

Multiplying Two Series

Similarly, to compute the Taylor series of $\frac{e^x}{1-x}$ around $x=0$, we can multiply the two elementary Taylor series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.

$$\begin{aligned} \frac{e^x}{1-x} &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{k=0}^{\infty} x^k \right) \\ &= \sum_{n,k=0}^{\infty} \frac{x^{n+k}}{n!} \end{aligned}$$

Defining a new index of summation $m = n+k$, we can write the series in order of increasing powers of x.

$$\begin{aligned} \frac{e^x}{1-x} &= \sum_{n,k=0}^{\infty} \frac{x^{n+k}}{n!} \\ &= \sum_{m,k=0}^{\infty} \frac{x^m}{(m-k)!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \frac{1}{(m-k)!} \right) x^m \end{aligned}$$

The interval of convergence of a product of series is *at least* the intersection of the series' individual intervals of convergence.

Here, recalling that the interval of convergence of the Taylor series of e^x is $(-\infty, \infty)$ and the interval of convergence of the Taylor series of $\frac{1}{1-x}$ is $(-1,1)$, we determine that the interval of convergence for the Taylor series of $\frac{e^x}{1-x}$ must be at least the intersection $(-\infty,\infty)\cap(-1,1)=(-1,1)$.

If we go through the trouble of performing a test of convergence, it's possible that we might find a larger interval of convergence -- but just based on the intervals of convergence of the two series being multiplied, we can say with certainty that the product converges for *at least* $-1<x<1$, without needing to perform any tests of convergence.

Adding Two Series

Sometimes, we can take advantage of the fact that it's easier to add or subtract series than to multiply series.

For example, to find the Taylor series of $e^x(1-e^x)$ around $x=0$, one option is to multiply the Taylor series of e^x and $1-e^x$.

However, an easier route is to simplify the expression to e^x-e^{2x}, and then subtract the Taylor series of e^{2x} from the Taylor series of e^x. To compute the Taylor series of e^{2x}, we can substitute $2x$ for x in the Taylor series of e^x.

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
e^{2x} &= \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \\
e^{2x} &= \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n
\end{aligned}
$$

Then, we can proceed with subtracting the Taylor series.

$$
\begin{aligned}
e^x(1-e^x) &= e^x - e^{2x} \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!} - \frac{2^n}{n!} x^n \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{n!} - \frac{2^n}{n!} \right) x^n \\
&= \sum_{n=0}^{\infty} \frac{1-2^n}{n!} x^n
\end{aligned}
$$

Again, the interval of convergence of a sum or difference of series is *at least* the intersection of the series' individual intervals of convergence.

The series for e^x converges for $-\infty < x < \infty$, so the series for e^{2x} converges for $-\infty < 2x < \infty$, which simplifies to $-\infty < x < \infty$. The intersection is given by

$(-\infty,\infty)\cap(-\infty,\infty)=(-\infty,\infty)$, so the interval of convergence of the series for $e^x - e^{2x}$ is *at least* $-\infty < x < \infty$.

Note that this interval contains all real numbers, so the interval can't get any bigger. Thus, the interval of convergence of the series for $e^x - e^{2x}$ is $-\infty < x < \infty$.

Using Differentiation and Integration

We can also use differentiation and integration to simplify the process of finding Taylor series.

For example, to find the Taylor series of $\sin^2 x$, one option is to multiply the series of $\sin x$ by itself -- but an easier option is to differentiate to yield a simpler result, then find the Taylor series of the simpler result, and then integrate the Taylor series to get back to the original function.

$$
\begin{aligned}
(\sin^2 x)' &= 2\sin x \cos x \\
(\sin^2 x)' &= \sin 2x \\
(\sin^2 x)' &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} \\
(\sin^2 x)' &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1} \\
\int (\sin^2 x)' \, dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1}}{(2n+1)!} x^{2n+1} \, dx \\
\sin^2 x &= C + \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1}}{(2n+1)!(2n+2)} x^{2n+2} \\
\sin^2 x &= C + \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1}}{(2n+2)!} x^{2n+2}
\end{aligned}
$$

To solve for the constant of integration, we can substitute $x = 0$.

$$
\begin{aligned}
\sin^2 0 &= C + \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1}}{(2n+2)!}(0)^{2n+2} \\
0 &= C + \sum_{n=0}^{\infty} 0 \\
0 &= C
\end{aligned}
$$

Thus, we have

$$
\sin^2 x = \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1}}{(2n+2)!} x^{2n+2} .
$$

Though not strictly necessary, we can clean up the series a bit by changing the index of summation to $k = n + 1$.

$$\sin^2 x = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}(2)^{2k-1}}{(2k)!} x^{2k}$$

Neither differentiating nor integrating a Taylor series changes its interval of convergence, so the interval of convergence of the series for $\sin^2 x$ is the same as the interval of convergence of the series for $\sin 2x$, which is $-\infty < x < \infty$.

Substitution

In the previous examples, we computed the series for $\sin 2x$ and e^{2x} by substituting $2x$ for x in the series for $\sin x$ and e^x. We can extend this idea to more clever substitutions.

For example, to compute the series of the function $\frac{1}{2+x^5}$, we can substitute $-\frac{x^5}{2}$ for x in the elementary series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

$$\begin{aligned} \frac{1}{2+x^5} &= \frac{1}{2} \cdot \frac{1}{1 - \left(-\frac{x^5}{2}\right)} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x^5}{2}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{5n} \end{aligned}$$

After substitution, the interval of convergence becomes $-1 < -\frac{x^5}{2} < 1$, which simplifies to $-\sqrt[5]{2} < x < \sqrt[5]{2}$.

Exercises

Compute the Taylor series for the following functions, centered at $x = 0$.

1) $f(x) = x\ln(1+x^2)$

2) $f(x) = \dfrac{1}{x^2}\arctan(2x^3)$

3) $f(x) = \dfrac{1}{1+2x}$

4) $f(x) = \cos\left(\sqrt{\pi x}\right)$

5) $f(x) = \ln(e+x)$

6) $f(x) = (\cos x + \sin x)(\cos x - \sin x)$

7) $f(x) = \sin^2(3x)$

8) $f(x) = \cos^2(\pi x)$

4.5 Solving Differential Equations with Taylor Series

Many differential equations don't have solutions that can be expressed in terms of finite combinations of familiar functions. However, we can often solve for the Taylor series of the solution.

Demonstration

For example, to solve the differential equation $y'' + xy' + xy = e^x$ we can substitute the Taylor series $y = \sum_{n=0}^{\infty} a_n x^n$ and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and solve for the coefficients a_n.

Differentiating, we have $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Substituting the derivatives in the differential equation, re-indexing so that all exponents are n, expressing all sums with the same starting index, and combining terms under a single sum, we condense the expression into a single polynomial.

$$\begin{aligned}
e^x &= y'' + xy' + xy \\
\sum_{n=0}^{\infty} \frac{x^n}{n!} &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=0}^{\infty} \frac{x^n}{n!} &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^{n} + \sum_{n=0}^{\infty} a_n x^{n+1} \\
\sum_{n=0}^{\infty} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=1}^{\infty} na_n x^{n} + \sum_{n=1}^{\infty} a_{n-1} x^{n} \\
1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} &= 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=1}^{\infty} na_n x^{n} + \sum_{n=1}^{\infty} a_{n-1} x^{n} \\
0 &= 2a_2 - 1 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + na_n + a_{n-1} - \frac{1}{n!} \right] x^n
\end{aligned}$$

For the expression to evaluate to 0, we must have $2a_2 - 1 = 0$ and $(n+2)(n+1)a_{n+2} + na_n + a_{n-1} - \frac{1}{n!} = 0$ for $n \geq 1$. So, we can choose a_0 and a_1 to be our constants $a_0 = C_1$ and $a_1 = C_2$, set $a_2 = \frac{1}{2}$, and express all other coefficients a_n for $n \geq 3$ in terms of the constants $a_0 = C_1$ and $a_1 = C_2$ through a recurrence:

$$\begin{aligned}
0 &= (n+2)(n+1)a_{n+2} + na_n + a_{n-1} - \frac{1}{n!} && (n \geq 1) \\
a_{n+2} &= \frac{1}{(n+2)(n+1)} \left(\frac{1}{n!} - na_n - a_{n-1} \right) && (n \geq 1) \\
a_n &= \frac{1}{n(n-1)} \left(\frac{1}{(n-2)!} - (n-2)a_{n-2} - a_{n-3} \right) && (n \geq 3)
\end{aligned}$$

Thus, our solution is given by $y = \sum_{n=0}^{\infty} a_n x^n$ where $a_0 = C_1$, $a_1 = C_2$, $a_2 = \frac{1}{2}$, and

$$a_n = \frac{1}{n(n-1)} \left(\frac{1}{(n-2)!} - (n-2)a_{n-2} - a_{n-3} \right) \qquad (n \geq 3).$$

As another example, we will solve the differential equation $y''' = y'y$ using the same process. We write the solution as the Taylor series $y = \sum_{n=0}^{\infty} a_n x^n$, substitute its derivatives into the equation, and simplify.

$$\begin{aligned}
y''' &= y'y \\
\sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3} &= \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
\sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3} &= \left(\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k \right) \left(\sum_{m=0}^{\infty} a_m x^m \right) \\
\sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3} &= \sum_{k,m=0}^{\infty} (k+1) a_{k+1} a_m x^{k+m} \\
\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} (k+1) a_{k+1} a_{n-k} \right] x^n
\end{aligned}$$

We can choose $a_0 = C_1$, $a_1 = C_2$, and $a_2 = C_3$ as our constants and express all other coefficients a_n for $n \geq 3$ in terms of these constants through a recurrence:

$$\begin{aligned}
(n+3)(n+2)(n+1)a_{n+3} &= \sum_{k=0}^{n}(k+1)a_{k+1}a_{n-k} && (n \geq 0) \\
a_{n+3} &= \frac{\sum_{k=0}^{n}(k+1)a_{k+1}a_{n-k}}{(n+3)(n+2)(n+1)} && (n \geq 0) \\
a_n &= \frac{\sum_{k=0}^{n-3}(k+1)a_{k+1}a_{n-3-k}}{n(n-1)(n-2)} && (n \geq 3)
\end{aligned}$$

Thus, our solution is given by $y = \sum_{n=0}^{\infty} a_n x^n$ where $a_0 = C_1$, $a_1 = C_2$, $a_2 = C_3$, and

$$a_n = \frac{\sum_{k=0}^{n-3}(k+1)a_{k+1}a_{n-3-k}}{n(n-1)(n-2)} \qquad (n \geq 3).$$

Exercises

Use Taylor series to solve the following differential equations.

1) $y''' + x^2 y = 1$

2) $y'' + xy = e^x$

3) $y''' = \dfrac{y'}{1-x}$

4) $y''' = xy' \ln(1+x)$

Solutions

to Exercises

Part 1

Chapter 1.1

1) -1

2) 10

3) 6

4) does not exist
left: 5, right: 3

5) -3

6) 1

7) 0

8) does not exist
left: 0, right: $\tan 1$

9) ∞

10) $-\infty$

11) ∞

12) does not exist
left: $-\infty$, right: ∞

13) ∞

14) $-\infty$

15) $\frac{1}{2}$

16) 0

17) does not exist

18) -2

19) ∞

20) $\frac{\log 2}{\log 3}$

21) 1

22) 7

23) $\frac{1}{4}$

24) ∞

25) 2

26) does not exist
left: ∞, right: $-\infty$

27) $\frac{1}{\sqrt{3}}$

28) $\frac{3}{4}$

29) $-\frac{2}{\sqrt{3}}$

30) ∞

Chapter 1.2

1) ∞

2) 0

3) e

4) 1

5) 0

6) 0

7) $\sqrt{3}$

8) 0

9) 2

10) 0

11) e^2

12) $e^{\frac{1}{3}}$

13) $e^{-\frac{10}{3}}$

14) $e^{\frac{e}{\pi}}$

Chapter 1.3

1) $f'(x) = 5$

2) $f'(x) = -3$

3) $f'(x) = 2x$

4) $f'(x) = 14x$

5) $f'(x) = 2x - 1$

6) $f'(x) = 3x^2$

7) $f'(x) = \frac{1}{\sqrt{x}}$

8) $f'(x) = \frac{\sqrt{3}}{2\sqrt{x}}$

9) $f'(x) = -\frac{1}{x^2}$

10) $f'(x) = -\frac{2}{(2x+1)^2}$

Chapter 1.4

1) $f'(x) = \frac{4}{3}x^{\frac{2}{3}}$

2) $f'(x) = -\frac{6}{x^7}$

3) $f'(x) = 6\sqrt{x}$

4) $f'(x) = -\frac{2}{5}x^{-\frac{1}{5}}$

5) $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

6) $f'(x) = \frac{1}{72}x^{-\frac{71}{72}} - 2x$

7) $f'(x) = -\frac{15}{2\sqrt{x^7}} - \frac{3}{2\sqrt{x}}$

8) $f'(x) = 6.2x^{2.1} + 51x^{101}$

9) $f'(x) = \sqrt{2}x^{\sqrt{2}-1} + 3x^{\sqrt{3}-1} + \frac{2}{x^{\sqrt{2}+1}}$

10) $f'(x) = -\frac{\pi e}{x^{\pi+1}} + \pi e x^{e-1} - e x^{\frac{e}{\pi}-1}$

Chapter 1.5

1) $f'(x) = 12x(2x^2 + 1)^2$

2) $f'(x) = 8(4x^3 - 2x)(x^4 - x^2)^7$

3) $f'(x) = \frac{x}{\sqrt{x^2+1}}$

4) $f'(x) = \frac{4x+2}{\sqrt{(2x+1)^2+3}}$

5) $f'(x) = \frac{-3}{(3x-2)^2}$

6) $f'(x) = \frac{-28x}{(x^2-3)^2}$

7) $f'(x) = \frac{-5x^4}{(1-x^5)^{\frac{3}{2}}}$

8) $f'(x) = \frac{1}{4\sqrt{x}\sqrt{\sqrt{x}+1}}$

9) $f'(x) = \frac{5\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}$

10) $f'(x) = \frac{5}{4}\left(x^{\frac{3}{2}} + x^{\frac{4}{3}}\right)^{\frac{1}{4}}\left(\frac{3}{2}x^{\frac{1}{2}} + \frac{4}{3}x^{\frac{1}{3}}\right)$

Chapter 1.6

1) $f'(x) = \frac{5x^2+12x+4}{2\sqrt{x}}$

2) $f'(x) = (x-3)(x+1)^2(5x-7)$

3) $f'(x) = x(x+1)^2(5x+2)$

4) $f'(x) = \frac{3}{2\sqrt{x}}(2x+3)^3(x-5)^2(14x^3+13x^2-30x-5)$

5) $f'(x) = \frac{1}{(x+2)^2}$

6) $f'(x) = \frac{1-x^2}{(x^2+1)^2}$

7) $f'(x) = -\frac{4x^2+22x+10}{(2x^2-5)^2}$

8) $f'(x) = \frac{-x^6+6x^4-3x^2+2}{(x^4-1)^2}$

9) $f'(x) = \frac{x^2(x+1)^3(4x^2-7x-3)}{(x-1)^4}$

10) $f'(x) = \frac{32x^{\frac{3}{2}}+12x^2-1}{\left(\sqrt{x}+2\right)^2\sqrt{x}}$

Chapter 1.7

1) $f'(x) = -\frac{1}{x(\ln x)^2}$

2) $f'(x) = \sec^2(x)e^{1+\tan x}$

3) $f'(x) = (\ln 2)2^{\sin x}\cos x$

4) $f'(x) = -\frac{1}{x^2+1}$

5) $f'(x) = -2x\tan x^2$

6) $f'(x) = \frac{2}{\ln 3}\cot 2x$

7) $f'(x) = -\frac{1}{x\sqrt{(\ln 5)^2-(\ln x)^2}}$

8) $f'(x) = \left(\sin x\cos^2 x\right)\left(2\cos^2 x - 3\sin^2 x\right)$

9) $f'(x) = \frac{2e^x}{(1-e^x)^2}$

10) $f'(x) = \frac{\arcsin x+\arccos x}{\sqrt{1-x^2}\arccos^2 x}$

11) $f'(x) = \frac{1+e^x-xe^x}{1+x^2+2e^x+e^{2x}}$

12) $f'(x) = \frac{\arcsin(x)(2\arccos x-\arcsin x)}{\sqrt{1-x^2}}$

13) $f'(x) = \frac{e^x(x\ln x-2)}{x(\ln x)^3}$

14) $f'(x) = \frac{\cot x-\ln(\sin x)}{e^x}$

Chapter 1.8

1) $x = -\sqrt{\frac{2}{3}} \quad \text{(max)}$

$x = \sqrt{\frac{2}{3}} \quad \text{(min)}$

2) $x = -\frac{15}{2} \quad \text{(min)}$

$x = 0 \quad \text{(saddle)}$

3) $x = \dfrac{1}{e}$ (min)

4) $x = \dfrac{5}{2}$ (max)

5) $x = -1 - \sqrt{2}$ (max)
$x = -1 + \sqrt{2}$ (min)

6) $x = -1$ (min)
$x = 1$ (min)

7) $x = -1$ (max)
$x = 0$ (saddle)
$x = 2$ (min)

8) $x = -3$ (min)
$x = 0$ (max)
$x = \dfrac{8}{3}$ (min)
$x = 3$ (max)

9) $x = -5$ (min)
$x = -2$ (max)
$x = 0$ (min)
$x = 5$ (max)

10) $x = \dfrac{1}{2}$ (min)
$x = 1$ (max)
$x = 10$ (min)

Chapter 1.9

1) 23.00

2) 3.88

3) -1.92

4) 0.77

5) 0.92

6) 1.39

7) 1.09

8) 2.99

9) 0.49

10) 1.00

Chapter 1.10

1) 1

2) 0

3) ∞

4) ∞

5) $\frac{1}{2}$

6) 0

7) 1

8) 1

9) $\frac{1}{\sqrt{e}}$

10) e

Part 2

Chapter 2.1

1) $\frac{1}{4}x^4 - x^3 - \frac{2}{x^3} + C$

2) $2x^4 - x + C$

3) $\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x + C$

4) $\frac{2}{3}x + \frac{5}{3}\ln x - \frac{1}{x} + C$

5) $\frac{2}{3}\tan 3x - 2\csc\frac{x}{2} + C$

6) $\frac{1}{3}\sec 3x + C$

7) $-12\cos\frac{x}{4} - \frac{4}{5}\tan 5x + C$

8) $-\frac{1}{3\pi}\csc \pi x + \frac{10}{3\pi}\cos \pi x + C$

9) $\frac{1}{4}e^{4x} + e^{-3x} - e^{-x} + C$

10) $-e^{-2x} + \frac{4}{3}e^{3x} + \frac{1}{5}e^{-5x} - 2x + C$

11) $2e^x + 2x + e^{-x} + \frac{1}{2}e^{-2x} + C$

12) $-\frac{2}{3}e^{-\frac{3}{2}x} - \frac{2}{3}e^{-3x} + C$

13) $\frac{1}{4}\arctan x - 5\arcsin x + C$

14) $2\arctan x + \frac{1}{\sqrt{3}}\arcsin x + C$

15) $\frac{1}{3}\arctan 3x - \frac{3}{4}\arcsin 4x + C$

16) $\frac{1}{5\sqrt{2}} \arctan \sqrt{2}x + \frac{1}{3} \arcsin \frac{3}{2}x + C$

Chapter 2.2

1) $\frac{93}{2}$

2) $\frac{1}{2} - \sqrt{3}$

3) $\frac{3}{5}\left(1 - e^{-10}\right)$

4) $\frac{52}{3}\pi^{\frac{3}{2}}$

5) $e - 1 - \frac{1}{\ln 2}$

6) $\frac{28}{3}$

7) $\frac{1}{3}\left(2e^3 - 3e^2 - 2e^{\frac{3}{2}}\right)$

8) $4\pi - 2 - \frac{\pi^3}{12}$

9) 3

10) $2\sqrt{2}$

11) $\frac{319}{2}$

12) 8

Chapter 2.3

1) $\frac{2}{3}(x+2)^{\frac{3}{2}} + C$

2) $\frac{1}{36}(4x+3)^9 + C$

3) $-\sqrt{1-x^2} + C$

4) $-\frac{1}{2(x^3-5)^2} + C$

5) $\frac{1}{3}\tan^3 x + C$

6) $2\sqrt{\sin x} + C$

7) $-\frac{1}{6(\sec^2 x+1)^3} + C$

8) $-\sin(\cos x) + C$

9) $\frac{1}{3}e^{x^3+1}+C$

10) $e^{-\cot x}+C$

11) $\frac{2}{3}e^{\sqrt{x^3-1}}+C$

12) $e^{e^x}+C$

13) $\arctan(e^x)+C$

14) $\arcsin(\ln x)+C$

15) $2\arcsin\sqrt{x}+C$

16) $\frac{1}{2}\arctan x^2+C$

Chapter 2.4

1) $(x^2-2x+2)e^x+C$

2) $\frac{1}{4}x^2(2\ln x-1)+C$

3) $(x+1)\sin x+\cos x+C$

4) $(2x^2-7x+7)e^x+C$

5) $\frac{1}{3}(x^3-1)e^{x^3}+C$

6) $\frac{1}{2}(\sin x-\cos x)e^x+C$

7) $\frac{1}{27}x^3\left[9(\ln x)^2-6\ln x+2\right]+C$

8) $\sin(e^x)-e^x\cos(e^x)+C$

9) $\frac{1}{2}\ln(x^2+1)+x\arctan\left(\frac{1}{x}\right)+C$

10) $\frac{1}{2}\cos x-\frac{1}{10}\cos 5x+C$

Chapter 2.5

1) $\frac{1}{4}$

2) ∞

3) $\frac{1}{5}$

4) $\frac{1}{81}$

5) $-\infty$

6) $\infty - \infty$ (indeterminate)

7) $\sqrt{\frac{17}{2}}$

8) $\frac{1}{\sqrt{7}}$

9) $\frac{1}{2}$

10) 2

11) 1

12) $\frac{\pi}{2}$

Part 3

Chapter 3.1

1) $y = 4x + C$

2) $y = \frac{1}{9}x^3 + C$

3) $y = -\frac{1}{2}\ln(x^2 - 1) + C$

4) $y = -\frac{1}{2}\cos x^2 + C$

5) $y = \pm\sqrt{C - 2\cos x}$

6) $y = -1 \pm \sqrt{C + 2\ln x}$

7) $y = xe^{C - e^x} - 1$

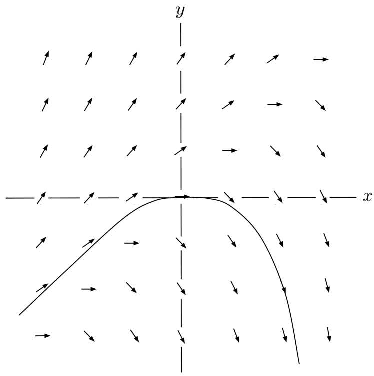
8) $y = \sin^{-1}\left(e^{\frac{1}{2}x^2 + C}\right)$

9) $y = e^x + e^{2x} + C_1x + C_2$

10) $y = \frac{1}{24}x^4 + \sin x + C_1x^2 + C_2x + C_3$

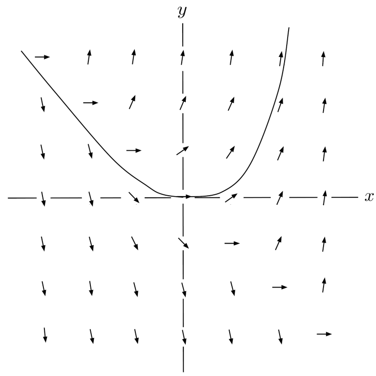
Chapter 3.2

1)



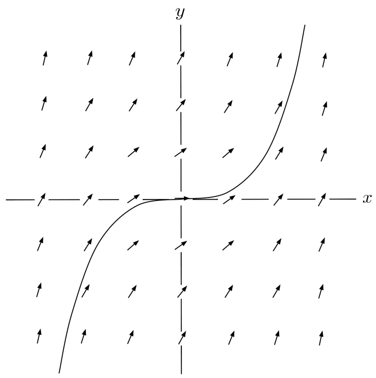
$$y(0.25) \approx 0$$
$$y(0.5) \approx -0.06$$
$$y(0.75) \approx -0.20$$
$$y(1) \approx -0.44$$

2)



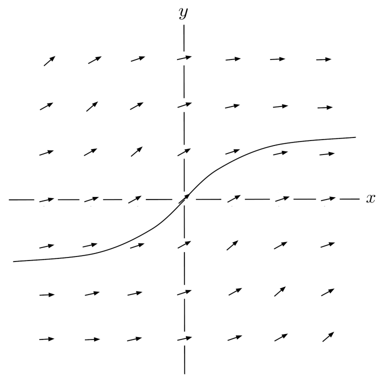
$$y(0.25) \approx 0$$
$$y(0.5) \approx 0.01$$
$$y(0.75) \approx 0.04$$
$$y(1) \approx 0.15$$

3)



$$y(0.25) \approx 0$$
$$y(0.5) \approx 0.06$$
$$y(0.75) \approx 0.19$$
$$y(1) \approx 0.38$$

4)



$$y(0.25) \approx 0.25$$
$$y(0.5) \approx 0.42$$
$$y(0.75) \approx 0.55$$
$$y(1) \approx 0.66$$

Chapter 3.3

1) $u = x + y$

$$y = \frac{1}{C - x} - x$$

2) $u = x - 2y$

$$y = Ce^x + \frac{1}{2}x$$

3) $u = x^2 - y^2$

$$y = \pm\sqrt{x^2 \pm \sqrt{C + 2x}}$$

4) $u = x^2 + y^3$

$$y = -\sqrt[3]{x^2 + \ln(C - x)}$$

5) $u = x^2 y$

$$y = \frac{C}{x^2} + \frac{1}{x}$$

6) $u = xy^2$

$$y = \pm\sqrt[4]{\frac{C}{x^2} + \frac{2}{x}}$$

Chapter 3.4

1) $y = C_1 e^{-4x} + C_2 e^{3x}$

2) $y = C_1 e^{-5x} + C_2 e^{-3x}$

3) $y = C_1 \cos 4x + C_2 \sin 4x$

4) $y = C_1 e^x + C_2 \cos\left(\sqrt{2}x\right) + C_3 \sin\left(\sqrt{2}x\right)$

5) $y = \left(C_1 \cos x + C_2 \sin x\right) e^{2x}$

6) $y = \left(C_1 \cos 4x + C_2 \sin 4x\right) e^{-x}$

7) $y = \left(C_1 + C_2 x\right) e^{2x}$

8) $y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 e^x + C_6 e^{-x}$

9) $y = C_1 + C_2 x + C_3 x^2 + C_4 \cos x + C_5 \sin x$

10) $y = C_1 + C_2 x + C_3 x^2 + C_4 e^x + C_5 e^{-2x}$

Chapter 3.5

1) $y = C_1 \cos x + C_2 \sin x + \frac{2}{13} e^{5x}$

2) $y = C_1 e^{-3x} + \frac{3\sin 2x - 2\cos 2x}{13}$

3) $y = C_1 e^{x} + C_2 - \frac{\sin(\pi x) + \pi\cos(\pi x)}{\pi + \pi^3}$

4) $y = C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x} + C_3 - \frac{1}{6}x^3 - x$

5) $y = C_1 e^{\frac{1}{2}x} - \frac{\sin x + 2\cos x}{5} + \frac{\cos(2x) - 4\sin(2x)}{17}$

6) $y = C_1 e^{-\frac{1}{2}x} + \frac{1}{3}e^{x} + \frac{3\sin x - 6\cos x}{5}$

7) $y = C_1 e^{\frac{3}{2}x} + C_2 e^{-\frac{3}{2}x} - \frac{2}{9}x^4 - \frac{23}{27}x^2 - \frac{1}{13}\cos(x+1) - \frac{184}{243}$

8) $y = C_1 e^{-x} + \frac{5\sin(5x) + \cos(5x)}{26} + \frac{\sin(2x+1) - 2\cos(2x+1)}{5} + 1$

Chapter 3.6

1) $y = \frac{C_1 + \sin x}{x} - \cos x$

2) $y = \frac{C_1 + 2x}{\ln x} - 2x - x\ln x$

3) $y = C_1 \csc x - \cot x$

4) $y = \frac{C_1 + \tan x}{x}$

5) $y = \frac{C_1 + \tan x}{x}$

6) $y = C_1 \sin x - \cos x$

Chapter 3.7

1) $y = e^{x}\left(C_1 + C_2 x - \ln x\right)$

2) $y = \frac{1}{4}e^{x}\left(C_1 + C_2 x - 3x^2 + 2x^2 \ln x\right)$

3) $y = \frac{1}{8}e^{2x}\left[C_1 + \ln\left(e^{-2x}+1\right)\right] - \frac{1}{8}e^{-2x}\left[C_1 + \ln\left(e^{2x}+1\right)\right] - \frac{1}{8}$

4) $y = \frac{1}{2}e^{-x}\left[C_1 + C_2 x - \ln(1+x^2) + 2x\arctan x\right]$

5) $y = C_1\cos x + C_2\sin x + \frac{1}{25}e^x\left[(10x-14)\sin x + (5x-2)\cos x\right]$

6) $y = C_1 + \frac{1}{2}e^x\left[C_2 + (-x^2+4x+1)\sin x + (-x^2-2x+5)\cos x\right]$

Part 4

Chapter 4.1

1)	$\frac{1}{2}$	2)	∞
3)	$\frac{1}{4}$	4)	2
5)	∞	6)	∞
7)	1	8)	$\frac{10}{11}$

Chapter 4.2

1)	diverges	2)	converges
3)	converges	4)	converges
5)	diverges	6)	diverges
7)	converges	8)	converges
9)	converges	10)	converges
11)	diverges	12)	converges
13)	diverges	14)	converges

15) diverges

16) converges

Chapter 4.3

1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$

$-1 < x \leq 1$

2) $\displaystyle\sum_{n=0}^{\infty} (-1)^n (1+n) x^n$

$-1 < x < 1$

3) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$-\infty < x < \infty$

4) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} (x-\pi)^{2n}$

$-\infty < x < \infty$

5) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

$-1 \leq x \leq 1$

6) $\displaystyle\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{(2n)!} (x+1)^n$

$-\infty < x < \infty$

7) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n$

$0 < x < 4$

8) $\displaystyle\sum_{n=0}^{\infty} -\frac{1}{10^{n+1}} (x+10)^n$

$-20 < x < 0$

Chapter 4.4

1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{2n+1}$

$-1 < x \leq 1$

2) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} x^{6n+1}$

$-\sqrt[3]{\frac{1}{2}} \leq x \leq \sqrt[3]{\frac{1}{2}}$

3) $\displaystyle\sum_{n=0}^{\infty}(-2)^n x^n$

$-\frac{1}{2} < x < \frac{1}{2}$

4) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n \pi^n}{(2n)!}x^n$

$0 \leq x \leq \infty$

5) $\displaystyle 1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{ne^n}x^n$

$-e < x \leq e$

6) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}x^{2n}$

$-\infty < x < \infty$

7) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}3^{2n}2^{2n-1}}{(2n)!}x^{2n}$

$-\infty < x < \infty$

8) $\displaystyle 1+\sum_{n=1}^{\infty}\frac{(-1)^n \pi^{2n}2^{2n-1}}{(2n)!}x^{2n}$

$-\infty < x < \infty$

Chapter 4.5

1) $a_0 = C_1,\ a_1 = C_2,\ a_2 = C_3,\ a_3 = \frac{1}{6},\ a_4 = 0$

$$a_n = -\frac{a_{n-5}}{n(n-1)(n-2)} \qquad (n \geq 5)$$

2) $a_0 = C_1,\ a_1 = C_2,\ a_2 = \frac{1}{2}$

$$a_n = \frac{1}{n(n-1)}\left[\frac{1}{(n-2)!} - a_{n-3}\right] \qquad (n \geq 3)$$

3) $a_0 = C_1,\ a_1 = C_2,\ a_2 = C_3$

$$a_n = \frac{\sum_{k=0}^{n-3}(k+1)a_{k+1}}{n(n-1)(n-2)} \qquad (n \geq 3)$$

4) $a_0 = C_1,\ a_1 = C_2,\ a_2 = C_3,\ a_3 = 0,\ a_4 = 0$

$$a_n = \frac{\sum_{k=1}^{n-4} \frac{(-1)^{n-k}}{n-k-3} a_k}{n(n-1)(n-2)} \qquad (n \geq 5)$$