Justin Math | **Algebra**

First Edition by Justin Skycak

First edition.

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Acknowledgements

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Part 1
Linear Equations and Systems

1.1 Solving Linear Equations

Loosely speaking, a **linear equation** is an equality statement containing only addition, subtraction, multiplication, and division. It does not need to include all of these operations, but it cannot include operations beyond them, such as exponentiation.

For example, these are linear equations:

$$5x + 9 = 5$$

$$14x - 6 = 3x + 2$$

$$-5x + 2 + 7x = 3x + 8 - x$$

On the other hand, these are not linear equations:

$$5x^2 + 9 = 5$$

$$14x - 6 = 3\sqrt{x} + 2$$

$$-5\sin(x) + 2 + 7x = 3|x| + 8$$

Solutions to Linear Equations

The **solution** of a linear equation is the value that we can substitute for the variable to make the equation true.

Most linear equations have a single solution. We can find the solution by performing operations on both sides of the equation, to isolate the variable.

Given equation	$5x + 8 = -2x + 22$
Add $2x$ to both sides	$7x + 8 = 22$
Subtract 8 from both sides	$7x = 14$
Divide both sides by 7	$x = 2$

To check our solution, we can substitute it in both sides of the equation and check that they evaluate to the same result:

$$\begin{aligned} 5(2) + 8 &= -2(2) + 22 \\ 10 + 8 &= -4 + 22 \\ 18 &= 18 \end{aligned}$$

Case of No Solutions

However, some linear equations have no solutions. When we try to solve these equations, the variable vanishes and we are left with an untrue statement.

Given equation	$3x + 1 = 2 + 3x$
Subtract $3x$ from both sides	$1 = 2$

This means that there is no number we can substitute for x to make the given equation true.

In fact, the right-hand side will always be 1 more than the left-hand side: the left-hand side says to multiply the input by 3 and add 1, while the left-hand side says to multiply the input by 3 and add 2.

Both sides multiply the input by 3, but then add different amounts! We can never hope to get the results to be the same.

Case of Infinitely Many Solutions

Even more interesting, some linear equations have infinitely many solutions. When we try to solve these equations, the variable still vanishes, but this time we are left with a true statement.

Given equation	$-2x + 1 = 1 - 2x$
Add $2x$ to both sides	$1 = 1$

In other words, any number we substitute for x will make the given equation true.

The left-hand side and the right-hand side will always come out to the same result: the left-hand side tells us to multiply the input by -2 and add 1, and the right-hand side tells us to multiply the input by 2 and then subtract it from 1. These are really just two ways of saying the same thing.

Exercises

Solve the following:

1) $2x + 3 = 11$

2) $-3x + 7 = 6 - 2x$

3) $-5x + 12 = 3x - 2 - x$

4) $-5x + 2 = 2x - 1 - 7x$

5) $-3x - 17 - 12x = -5x + 13$

6) $18 - x + 1 = 10x + 19 - 11x$

7) $8(x+4) + 12(x-4) = 84$

8) $4(2x+3) - x = 3(2x+4) + x$

9) $5\left[4(3x+1) - 3(3x+2)\right] + 25 = -75$

10) $3\left[2(x+1) - (x+2)\right] = 3x + 1$

1.2 Slope-Intercept Form

Before, we were solving linear equations in one variable. Now, let's consider linear equations in two variables. A few examples are shown below:

$$3x + 2y = 1$$

$$8y - 5 + 3x = 10x$$

$$15y - 2x + 3 = 3(x + y) - 4$$

Solutions to Two-Variable Equations

The solution to a two-variable linear equation is no longer just the number(s) that we can substitute for x to make the equation true, but rather the pair(s) (x, y) that we can substitute for x and y to make the equation true.

Two-variable linear equations usually have infinitely many solutions, because we are usually able to solve for one variable in terms of the other.

Given equation	$10x - 5y + 15 = 35$
Subtract 15 from both sides	$10x - 5y = 20$
Subtract $10x$ from both sides	$-5y = -10x + 20$
Divide both sides by -5	$y = 2x - 4$

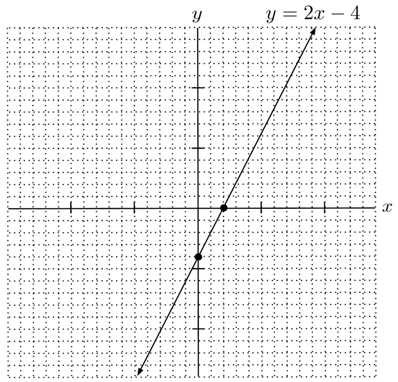
If we choose $x = 2$, then we can make the given equation true by choosing $y = 2(2) - 4 = 0$. If we choose $x = 10$, then we can make the given equation true by choosing $y = 2(10) - 4 = 16$. Whatever value we choose for x, we can make the equation true by choosing y as twice that value, minus 4.

However, although there are infinitely many solutions to the equation, that doesn't mean that any random pair we pick will be a solution. For example, if we try the pair $(x, y) = (2, 1)$, then the left-hand side comes out to $10(2) - 5(1) + 15 = 30$, not 35.

Graphing

To really see what's going on, it helps to plot the solutions on a graph. In fact, linear equations are called linear because when we plot them on a graph, they form a straight line

To plot all the solutions of $y = 2x - 4$ on the graph below, we plot two solutions and draw a line through them. We already saw that one solution was $(2, 0)$, and when we substitute $x = 0$ we get $y = 2(0) - 4 = -4$, so another solution is $(0, -4)$.



Any point that is on the line is a solution of the original equation. For example, we see that the line passes through the point $(4, 4)$ -- and indeed, substituting $x = 4$ and $y = 4$ makes the original equation true.

$$\begin{aligned} 10x - 5y + 15 &= 35 \\ 10(4) - 5(4) + 15 &= 35 \\ 40 - 20 + 15 &= 35 \\ 35 &= 35 \end{aligned}$$

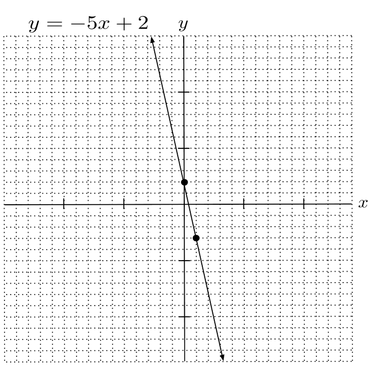
Slope-Intercept Form

In general, when we solve for y in a linear equation of two variables, we end up with a result in the form $y = mx + b$ where m and b are constants (provided y doesn't vanish). This is called **slope-intercept** form, and the constants m and b are called the **slope** and **y-intercept** of the line, respectively.

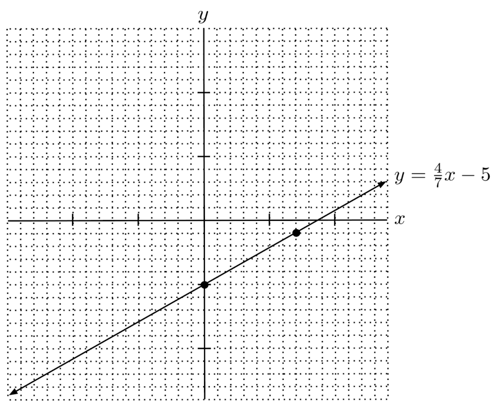
The y-intercept takes its name from the fact that the line crosses the y-axis at b. For example, the graph of $y = 2x - 4$ shown earlier crossed the y-axis at -4. This pattern is true in general because the pair $(0, b)$ is a solution of the equation $y = mx + b$: when we substitute $x = 0$, we find $y = m(0) + b = b$.

The slope takes its name from the fact that m controls how steep the line is: for every unit the line travels right, it travels m units up (or down, if m is negative). For example, in the graph of $y = 2x - 4$, if we start at the point $(2, 0)$ and travel 1 unit right and 2 units up, we arrive at the point $(3, 2)$, which is also on the line.

To graph a line $y = mx + b$ in slope-intercept form, it is easiest to start by plotting the intercept $(0, b)$. Then, we can pick another point by going right 1 unit and up m units. For example, to plot the line $y = -5x + 2$, we can start at the intercept $(0, 2)$, and since the slope is -5, we will go right 1 unit and down 5 units to arrive at a second point $(1, -3)$. Then, we can connect these two points with a line.



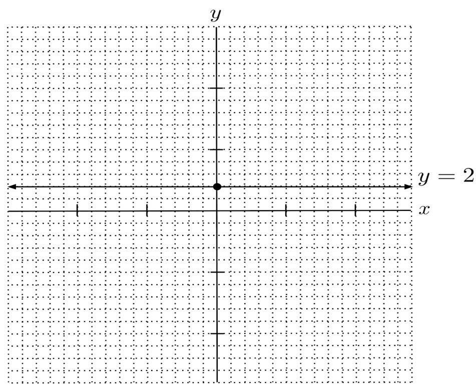
When we have a fractional slope, such as in the line $y = \frac{4}{7}x - 5$, it is easier to go right 7 units and up 4 units, instead of going right 1 unit and up $\frac{4}{7}$ of a unit. We're just repeating the process 7 times, for a total distance right of $(1)(7) = 7$ and a total distance up of $\left(\frac{4}{7}\right)(7) = 4$. The resulting line is shown in the graph below.



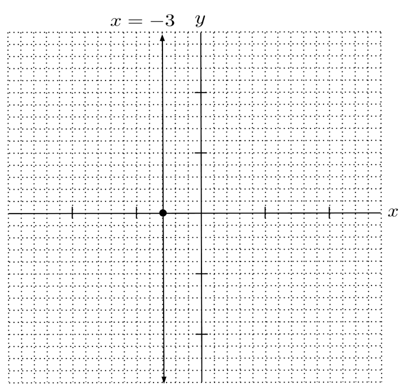
Horizontal and Vertical Lines

If the x term vanishes when we solve for y, such as in the line $y + x = 2 + x$ which simplifies to $y = 2$, then we can interpret the slope as being 0 because the line can be written $y = 0x + 2$. The resulting line has a y-intercept $(0, 2)$ and is horizontal because for every unit it goes to the right, it goes 0 units up.

Perhaps an easier way to think about it, though, is that the solution is just all the points that have a y-coordinate of 2, regardless of their x-coordinates.



On the other hand, if y vanishes when we solve, such as in the line $y - x = y + 3$ which simplifies to $x = -3$, then we have a vertical line that passes through all the points having an x-coordinate of -3, regardless of their y-coordinate.

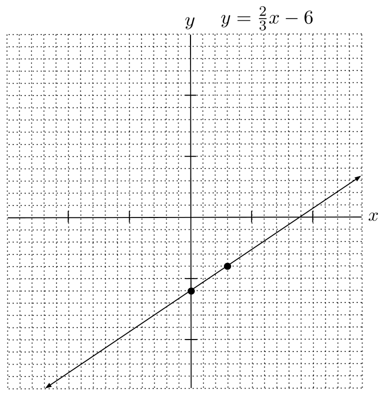


Finding the Equation from a Graph

Now, let's think in reverse: if we draw a particular line, how can we come up with its equation?

If we know the y-intercept and slope of the line, then it's easy -- we just substitute the slope for m and the y-intercept for b in the equation $y = mx + b$.

For example, in the line below, we see that the y-intercept is -6, and when we go right 3, we go up 2, so the slope is $\frac{2}{3}$. The equation of the line, then, is $y = \frac{2}{3}x - 6$.



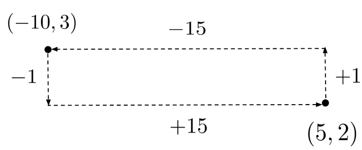
But what if we aren't given the slope and y-intercept, or even a picture of the line, and we want to write the equation of the line based on only two points it passes through?

It's straightforward to compute the slope based on the two points -- we just need to find the *rise*, or the change in y, and divide it by the *run*, or the change in x.

For example, if the points are $(-10,3)$ and $(5,2)$, then we can compute the rise as $2-3=-1$ and the run as $5-(-10)=15$, resulting in a slope of $m=\frac{-1}{15}=-\frac{1}{15}$.

Or, we can compute the rise as $3-2=1$ and the run as $-10-5=-15$, still resulting in a slope of $m=\frac{1}{-15}=-\frac{1}{15}$.

Either way, we get the same slope.



Substituting for m in the equation $y=mx+b$, we reach

$$y=-\frac{1}{15}x+b.$$

It remains to find the y-intercept, b. We can do this by substituting for x and y using the coordinates of one of the points that we know needs to be on the line, say, $(5,2)$.

$$\begin{aligned}
y &= -\frac{1}{15}x + b \\
2 &= -\frac{1}{15}(5) + b \\
2 &= -\frac{1}{3} + b \\
b &= \frac{7}{3}
\end{aligned}$$

It really doesn't matter which point we use -- even if we used the other point, $(-10, 3)$, we would get the same result for b.

$$\begin{aligned}
y &= -\frac{1}{15}x + b \\
3 &= -\frac{1}{15}(-10) + b \\
3 &= \frac{2}{3} + b \\
b &= \frac{7}{3}
\end{aligned}$$

Now that we know the y-intercept is $b = \frac{7}{3}$, we can write the final equation of the line:

$$y = -\tfrac{1}{15}x + \tfrac{7}{3}$$

Exercises

Graph the following linear equations.

1) $y = 3x - 5$

2) $y = -4x + 6$

3) $y = \frac{3}{5}x + 2$

4) $y = 4 - \frac{2}{7}x$

5) $8y = 16 - 4x$

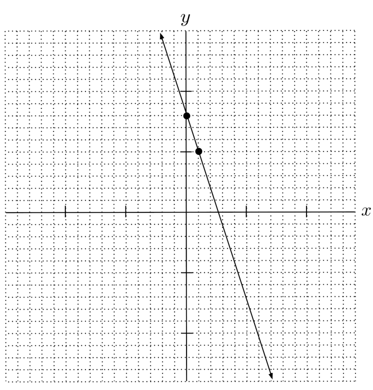
6) $2x = 2(x + y) + 1$

7) $4x + 10 = 2(x + 2y)$

8) $2(1 - y) = 8(x - 2)$

Write the equation of the line in slope-intercept form.

9)



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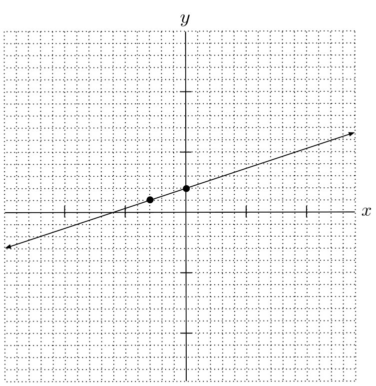

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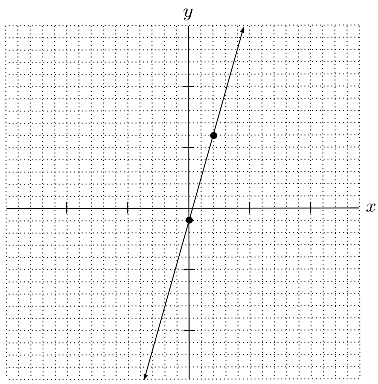

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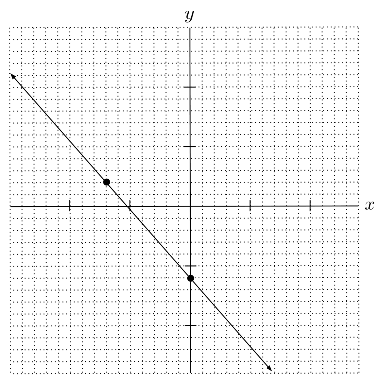


Write the slope-intercept equation of the line that goes through the given point, with the given slope.

13) $(0,-2)$ $m=3$

14) $(1,4)$ $m=-2$

15) $(-2,-4)$ $m=\frac{1}{2}$

16) $(1,-6)$ $m=-\frac{5}{7}$

Write the slope-intercept equation of the line that goes through the given points.

17) $(0,3)$ $(2,0)$

18) $(1,1)$ $(-3,2)$

19) $(-4,-5)$ $(6,7)$

20) $(-8,5)$ $(2,5)$

1.3 Point-Slope Form

Suppose we want to write the equation of a line with a given slope $m = 2$, through a particular point $(3, 5)$. In the previous chapter, we substituted the given information into a slope-intercept equation form $y = mx + b$, solved for b, and rewrote the slope-intercept form with m and b substituted so that x and y were the only variables.

Slope-intercept equation form	$y = mx + b$
Substitute the given slope $m = 2$	$y = 2x + b$
Substitute the given point $(3, 5)$	$5 = 2(3) + b$
Solve for b	$b = -1$
Final equation	$y = 2x - 1$

However, there is an alternative form, **point-slope form**, that makes it even easier to write the equation of a line if we know the slope m and a point (x_0, y_0) on the line. It is given by

$$y - y_0 = m(x - x_0).$$

If we know that our desired line has slope $m = 2$ and passes through the point $(x_0, y_0) = (3, 5)$, then we can substitute directly into point-slope form without performing any additional computations:

$$y - 5 = 2(x - 3)$$

This is an accepted form of the equation for a line, so we don't need to simplify it at all unless we're asked to do so.

But even if we actually need to find the line in slope-intercept form, it's still advantageous to begin with point-slope form, because all we have to do is distribute the 2 and add 5 to get to slope-intercept form.

Point-slope form	$y - 5 = 2(x - 3)$
Distribute the 2	$y - 5 = 2x - 6$
Add 5 to both sides to reach slope-intercept form	$y = 2x - 1$

Derivation

The point-slope formula is easy to remember, too, because it just says that the slope between any point (x, y) and the reference point (x_0, y_0) needs to be equal to the given slope m.

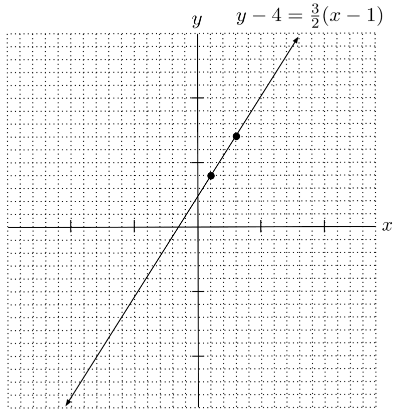
Moving from (x_0, y_0) to (x, y), the amount we go up is $y - y_0$, and the amount we go over horizontally is $x - x_0$, so the slope is just $\frac{y-y_0}{x-x_0}$. Equating this to m and multiplying to get rid of the fraction, we reach point-slope form!

Slope must equal m	$\frac{y-y_0}{x-x_0} = m$
Multiply both sides by $x - x_0$ to reach point-slope form	$y - y_0 = m(x - x_0)$

Graphing

To graph a line whose equation is given in point-slope form, we perform the same process as we do to graph a line that is in slope-intercept form, except we start at the reference point rather than at the y-intercept.

For example, consider the line $y - 4 = \frac{3}{2}(x - 1)$, for which the reference point is $(1, 4)$ and the slope is $\frac{3}{2}$. To graph this line, we start at $(1, 4)$, go up 3 and over 2 to the point $(3, 7)$, and draw a line through the two points.



Final Remark

One thing to watch out for in point-slope form: be careful about negatives.

For example, the point-slope form of a line with slope 2 that goes through the point $(-3,-5)$ is NOT given by $y-5=2(x-3)$. This is the line that goes through the point $(3,5)$, not $(-3,-5)$.

The line that goes through $(-3,-5)$ actually involves addition rather than subtraction, because the negatives cancel the subtraction in the original formula for point-slope form.

Point-slope formula	$y - y_0 = m(x - x_0)$
Substitute slope 2 and point $(-3, -5)$	$y - (-5) = 2(x - (-3))$
Negatives cancel	$y + 5 = 2(x + 3)$

Exercises

Write the point-slope equation of the line that goes through the given point, with the given slope.

1) $(1, 5)$ $\quad m = 2$

2) $(-2, 3)$ $\quad m = 8$

3) $\left(\frac{1}{2}, -2\right)$ $\quad m = \frac{3}{8}$

4) $\left(-\frac{4}{7}, \frac{8}{13}\right)$ $\quad m = -\frac{12}{5}$

Write the point-slope equation of the line that goes through the given points.

5) $(2, -1)$ $\quad (1, 1)$

6) $(1, 8)$ $\quad (-4, -7)$

7) $\left(\frac{1}{3}, 3\right)$ $\quad (1, 4)$

8) $\left(-\frac{3}{4}, \frac{1}{2}\right)$ $\quad \left(\frac{1}{2}, \frac{3}{4}\right)$

Graph the following lines.

9) $y - 2 = 3(x - 4)$

10) $y + 7 = -2(x - 2)$

11) $y - \frac{1}{2} = \frac{1}{3}(x + 1)$

12) $y + \frac{5}{2} = -\frac{2}{5}\left(x - \frac{1}{4}\right)$

1.4 Standard Form

The **standard form** of a linear equation is $ax + by = c$, where a, b, and c are all integers and a is nonnegative.

For example, we can convert the equation $y = \frac{3}{5}x + \frac{10}{3}$ to standard form by moving x and y to the same side and multiplying to cancel out any fractions.

Given equation	$y = \frac{3}{5}x + \frac{10}{3}$
Subtract $\frac{3}{5}x$ from both sides	$-\frac{3}{5}x + y = \frac{10}{3}$
Multiply both sides by 15, the least common multiple of 5 and 3	$-9x + 15y = 50$
Multiply both sides by -1 to make the x coefficient positive	$9x - 15y = -50$

Finding the Intercepts

Standard form makes it easy to see the intercepts of the line: to get the x-intercept in $ax + by = c$, we divide the constant c by the x-coefficient a, and to get the y-intercept, we divide the constant c by the y-coefficient b.

For example, the x-coefficient of $9x - 15y = -50$ is $-\frac{50}{9}$, and the y-coefficient is $\frac{-50}{-15}$ which simplifies to $\frac{10}{3}$.

This trick for finding the intercepts works because finding the intercept of a particular variable involves substituting 0 for the other variable. The x-intercept occurs at some point $(x, 0)$ where y is 0, so to solve for the x-intercept, we can substitute 0 for y and solve for x.

Given equation	$ax + by = c$
Substitute 0 for y	$ax + b(0) = c$
Simplify	$ax = c$
Divide by a	$x = \frac{c}{a}$

Likewise, the y-intercept occurs at some point $(0, y)$ where x is 0, so to solve for the y-intercept, we can substitute 0 for x and solve for y.

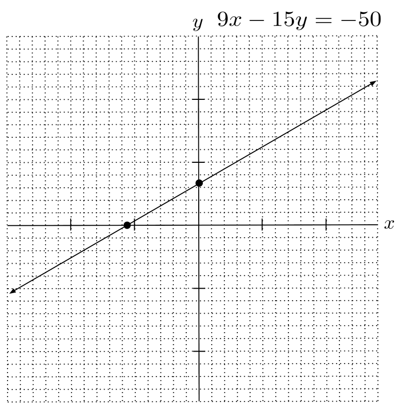
Given equation	$ax + by = c$
Substitute 0 for x	$a(0) + by = c$
Simplify	$by = c$
Divide by b	$y = \frac{c}{b}$

Graphing

To plot the line, then, all we have to do is mark the intercepts and then draw a line through them.

For example, in the line $9x - 15y = -50$, we computed the x-intercept as $-\frac{50}{9}$, or $-5\frac{5}{9}$, and the y-intercept as $\frac{10}{3}$, or $3\frac{1}{3}$.

To graph the line, we just need to plot the intercepts $\left(-5\frac{5}{9}, 0\right)$ and $\left(0, 3\frac{1}{3}\right)$ and draw a line through them.



Exercises

Write the equation in standard form. (It may already be in standard form.)

1) $y = \frac{3}{4}x - 1$

2) $-2x + 3y = 4$

3) $\frac{1}{3}x - y = 2$

4) $5x - 4y = 1$

5) $y + x = 2$

6) $4x + y = \frac{1}{3}$

Graph the following by drawing a line through the intercepts.

7) $3x + 2y = 9$

8) $x - 2y = 4$

9) $8x + y = -8$

10) $2x - 5y = -10$

1.5 Linear Systems

A **linear system** consists of multiple linear equations, and the solution of a linear system consists of the pairs that satisfy all of the equations.

For example, the solution to the linear system

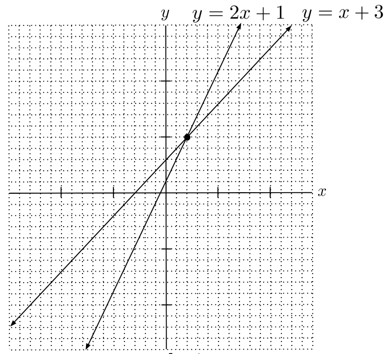
$$\begin{cases} y = 2x + 1 \\ y = x + 3 \end{cases}$$

is $(2, 5)$ because substituting 2 for x and 5 for y makes both equations true.

Graphical Interpretation

Graphically, we can think of a linear system as being a set of two lines, and their solution as the point where they intersect.

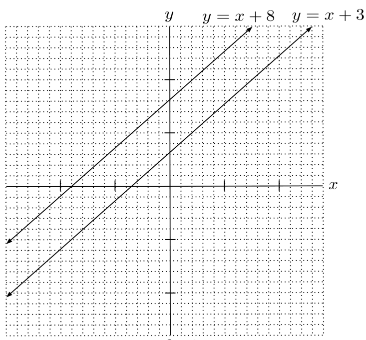
The intersection point is the solution because it is on both lines, meaning it makes both equations true.



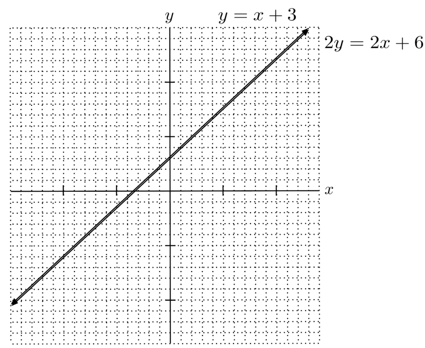
one solution

Usually, two lines will intersect in exactly one point, and thus the system will have a single solution. However, when the two lines are **parallel**, meaning that they have the same slope, the lines will never intersect, unless they are actually the same line.

If the system consists of two different parallel lines, then it will have no solution because there are no intersection points. But if the system consists of two lines that are actually the same, then the system will have infinitely many solutions because every point on the line is a solution.



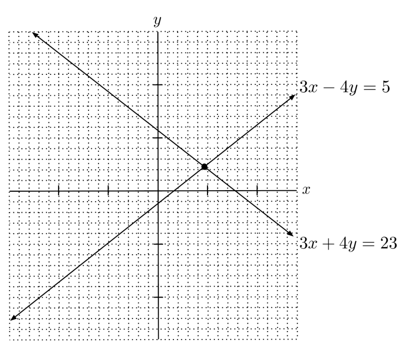
no solution



infinitely many solutions

We can sometimes tell the solution of a system by graphing the equations and looking for the point where they intersect. However, when the lines intersect at a point that doesn't coincide with grid lines on the graph, it can be difficult to identify the exact coordinates of the intersection point.

For example, can you identify the point of intersection below? If you think you can, would you bet your life on it?



Substitution

There is another method for solving a system of linear equations, called the method of **substitution**, which makes it possible to solve a linear system without graphing it.

To perform substitution, we create a third equation by solving for a particular variable in the first and second equations and setting the results equal to each other.

Since the third equation has a single variable, we can solve for the numeric value of that variable, and then use it to find the numeric value of the other variable.

Given system	$\begin{cases} 3x + 4y = 23 \\ 3x - 4y = 5 \end{cases}$
Solve for y	$\begin{cases} 4y = -3x + 23 \\ -4y = -3x + 5 \end{cases}$
	$\begin{cases} y = -\frac{3}{4}x + \frac{23}{4} \\ y = \frac{3}{4}x - \frac{5}{4} \end{cases}$
Set the results equal to each other	$-\frac{3}{4}x + \frac{23}{4} = \frac{3}{4}x - \frac{5}{4}$
Solve for x	$\frac{28}{4} = \frac{6}{4}x$

	$x = \frac{14}{3}$
Substitute $x = \frac{14}{3}$ in equation for y	$y = \frac{3}{4}\left(\frac{14}{3}\right) - \frac{5}{4}$
	$y = \frac{9}{4}$
Final solution	$\left(\frac{14}{3}, \frac{9}{4}\right)$

To perform substitution even more quickly, instead of solving for a particular variable in both equations, we can solve for a particular variable in just one of the equations and then substitute the resulting expression where the particular variable occurs in the other equation.

Given system	$\begin{cases} 3x + 4y = 23 \\ 3x - 4y = 5 \end{cases}$
Solve for y in bottom equation	$-4y = -3x + 5$
	$y = \frac{3}{4}x - \frac{5}{4}$
Substitute into top equation	$3x + 4\left(\frac{3}{4}x - \frac{5}{4}\right) = 23$
Solve for x	$3x + 3x - 5 = 23$
	$x = \frac{14}{3}$
Substitute $x = \frac{14}{3}$ in equation for y	$y = \frac{3}{4}\left(\frac{14}{3}\right) - \frac{5}{4}$

	$y = \frac{9}{4}$
Final solution	$\left(\frac{14}{3}, \frac{9}{4}\right)$

Remember that some systems have no solutions, and other solutions have infinite solutions -- so it shouldn't throw us off if the third equation created by substitution has no solutions or infinite solutions.

Elimination

An *even faster* way to solve some linear equations is the method of **elimination**. The method of elimination also creates a third equation in a single variable, but it does so by adding multiples of the two original equations to cancel out one of the variables.

Given system	$\begin{cases} 3x + 4y = 23 \\ 3x - 4y = 5 \end{cases}$
Add the two equations	$3x + 3x + 4y - 4y = 23 + 5$
y cancels	$6x = 28$
Solve for x	$x = \frac{14}{3}$
Substitute $x = \frac{14}{3}$ in top equation	$3\left(\frac{14}{3}\right) + 4y = 23$

Solve for y	$14 + 4y = 23$
	$y = \frac{9}{4}$
Final solution	$\left(\frac{14}{3}, \frac{9}{4}\right)$

In the previous example, one of the variables cancelled when we added the two equations. Other times, though, no variable will cancel right away, and we will first need to multiply one of the equations by a number so that a variable will cancel when we add the equations.

Given system	$\begin{cases} x + 2y = 4 \\ 3x + 4y = 1 \end{cases}$
Multiply top equation by -2	$\begin{cases} -2x - 4y = -8 \\ 3x + 4y = 1 \end{cases}$
Add the two equations to cancel y	$x = -7$

Other times still, we may need to multiply both equations by a different number to cancel a variable. (We can just take the least common multiple -- the same trick we use to add fractions with different denominators.)

Given system	$\begin{cases} 2x+3y=1 \\ 3x+5y=2 \end{cases}$
Multiply top equation by -3 Multiply bottom equation by 2 (Least common multiple is 6)	$\begin{cases} -6x-9y=-3 \\ 6x+10y=4 \end{cases}$
Add the two equations to cancel x	$y=1$

Again, since some systems have no solutions, and other solutions have infinite solutions, we should not be worried if the third equation created by elimination simplifies to a never-true statement like $2=1$ (no solutions) or an always-true statement like $1=1$ (infinite solutions).

Exercises

Solve by substitution or elimination.

1) $\begin{cases} y=2x-3 \\ y=x-1 \end{cases}$

2) $\begin{cases} y=5x+1 \\ y=5x-1 \end{cases}$

3) $\begin{cases} 2x+y=5 \\ x-y=4 \end{cases}$

4) $\begin{cases} 5x+2y=9 \\ 2x-y=-9 \end{cases}$

5) $\begin{cases} 3x + 4y = 2 \\ 15x + 20y = 10 \end{cases}$

6) $\begin{cases} y + 2x = 3 \\ x + 2y = 3 \end{cases}$

7) $\begin{cases} 5x - 3y = 4 \\ 40y + 10x = 31 \end{cases}$

8) $\begin{cases} 10x = 2y + 3 \\ y = 5x \end{cases}$

9) $\begin{cases} 13 = x + 4y \\ x - y = 5 \end{cases}$

10) $\begin{cases} 5 + 2y = 7x \\ x + 3y = 10 \end{cases}$

Part 2
Quadratic Equations

2.1 Standard Form

Quadratic equations are similar to linear equations, except that they contain squares of a single variable.

For example, the equations below are quadratic equations:

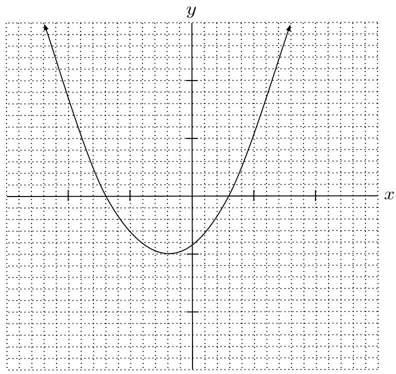
$$x^2 - 2x = 5 \qquad 5 = 4x^2 \qquad y + 2x = x^2$$

On the other hand, the equations below are not quadratic equations. (A quadratic equation must contain the square of one variable, but cannot contain squares of multiple different variables, and cannot contain other operations not found in linear equations, such as square roots.)

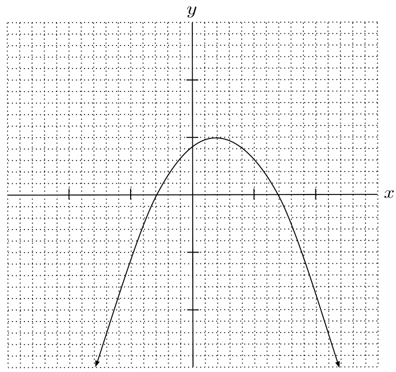
$$x + 2 = 3 \qquad x^2 - 5 = 2\sqrt{x} + x \qquad x^2 + y^2 = 4$$

Graphing

As a consequence of the squared variable, the shape of the graph of a two-variable quadratic equation is a parabola.



upward parabola



downward parabola

To tell whether the graph of a quadratic equation is an upward or downward parabola, it is helpful to arrange the quadratic equation into **standard form**, which is given by

$$y = ax^2 + bx + c$$

where a, b, and c are constants and called **coefficients**. The coefficient on the x^2 term, which is given by a, is often called the leading coefficient because it is the leftmost coefficient when terms in the standard equation are ordered properly.

Keep in mind that some coefficients may be zero -- for example, the quadratic equation $y = 3x^2 + 1$ has $b = 0$ because it can be written as $y = 3x^2 + 0x + 1$.

If the leading coefficient, a, is positive, then the parabola opens upward. Otherwise, if the leading coefficient is negative, then the parabola opens downward.

To remember this, you might think of a *positive* leading coefficient causing the parabola to *smile*, and a *negative* leading coefficient causing the parabola to *frown*.

$y = 2x^2 + x - 3$	Opens upward because leading coefficient (2) is positive
$y = -5x^2 + 2$	Opens downward because leading coefficient (-5) is negative

Sometimes, we may have to rearrange a quadratic equation into standard form.

Given Equation	$y = 1 + x^2 - 4x$	$-3 = y + 2x^2 + 6x$
Standard Form	$y = x^2 - 4x + 1$	$y = -2x^2 - 6x + 3$
Leading Coefficient	1	-2
Opening Direction	up	down

Vertex of a Parabola

The standard form of a quadratic equation can also tell us about the parabola's **vertex**, or turning point.

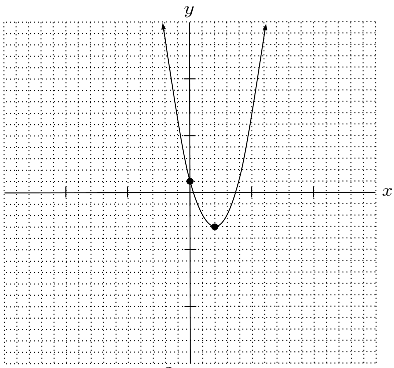
For a quadratic equation in the form $y = ax^2 + bx + c$, the x-coordinate of the vertex is given by $-\frac{b}{2a}$.

To find the y-coordinate of the vertex, we can substitute the x-coordinate of the vertex into the quadratic equation and evaluate.

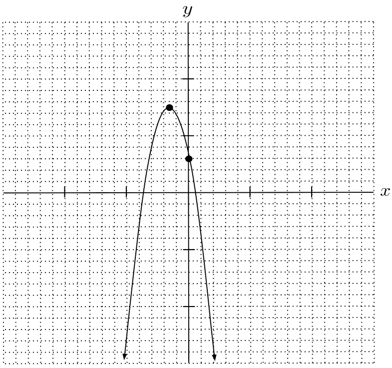
Standard Form	$y = x^2 - 4x + 1$	$y = -2x^2 - 6x + 3$
X-Coord of Vertex	$-\frac{-4}{2(1)}$	$-\frac{-6}{2(-2)}$
	2	$-\frac{3}{2}$
Y-Coord of Vertex	$2^2 - 4(2) + 1$	$-2\left(-\frac{3}{2}\right)^2 - 6\left(-\frac{3}{2}\right) + 3$
	-3	$\frac{15}{2}$
Vertex	$(2, -3)$	$\left(-\frac{3}{2}, \frac{15}{2}\right)$

With a parabola's vertex and direction of opening, we can draw a decent sketch of the graph.

To make our graph a little more accurate, we can also make sure it has the correct y-intercept. Since we set $x = 0$ to find the y-intercept, the y-intercept of $y = ax^2 + bx + c$ is always given by $a(0)^2 + b(0) + c$, which evaluates simply to c.



$y = x^2 - 4x + 1$



$y = -2x^2 - 6x + 3$

Exercises

For the following quadratic equations:

a. Write the quadratic equation in standard form.
b. Using the standard form, tell whether the parabola opens upward or downward, and find the vertex and y-intercept.
c. Finally, using the parabola's vertex, opening direction, and y-intercept, draw a rough sketch of the graph of the equation. (If the vertex and the y-intercept are the same, choose some other point.)

1) $y = 1 + x^2$

2) $y = 2x + x^2 - 3$

3) $y = 2(2x - 1) - x^2$

4) $y = 2x(x + 4)$

5) $y = 9x - 3(x^2 + 1) - 1$

6) $y = 4 - 5x(x + 1)$

7) $y - 6x - 2 = x(x+1)$

8) $10x + y = 10x(x+2)$

9) $3(y - x^2 - 4x) = 13 + 2y$

10) $x(3x+4) = 1 - y$

2.2 Factoring

Factoring is a method for solving quadratic equations. It involves converting the quadratic equation to standard form, then **factoring** it into a product of two linear terms (called **factors**), and finally solving for the variable values that make either factor equal to 0.

Original quadratic equation	$2+x^2=-3x$
Convert to standard form	$x^2+3x+2=0$
Factor	$(x+1)(x+2)=0$
Set each factor to 0	$x+1=0$ or $x+2=0$
Solve	$x=-1$ or $x=-2$

When we factor, we are rearranging the equation to say that the product of two numbers is 0. The equation is solved when either number is 0, because any number multiplied by 0 is 0.

How to Factor

Factoring is easiest in hindsight. Multiplying through, we see that the factored form is equivalent to the standard form:

$$\begin{aligned} (x+1)(x+2) &= 0 \\ x(x+2)+1(x+2) &= 0 \\ x^2+2x+1x+2 &= 0 \\ x^2+3x+2 &= 0 \end{aligned}$$

But how can we know this to begin with? In other words, if we want to factor an expression x^2+bx+c into the form $(x+m)(x+n)$, how do we know what m and n are?

Here's the trick: m and n need to multiply to c and add to b.

To factor the expression x^2+5x+4, we need to find two numbers that multiply to 4 and add to 5. Although 2 and 2 multiply to 4, they don't add to 5. But 1 and 4 multiply to 4 AND add to 5, so they work! The factored form is then $(x+1)(x+4)$.

Even with negatives, the method is still the same: to factor the expression x^2-2x-3, we need to find two numbers that multiply to -3 and add to -2. Although -1 and 3 multiply to -3, they don't add to -2. But 1 and -3 multiply to -3 AND add to -2, so they work! The factored form is then $(x+1)(x-3)$.

Case of Many Potential Factors

Factoring can become a little tricky when c has a lot of factors. In such cases, it can be helpful to make a factor table.

For example, to factor $x^2 + 26x + 144$, we can list out the factors of 144 and find which pair adds to 26. Since this pair is 8 and 18, the expression factors to $(x+8)(x+18)$.

Factor Pair	Sum
1 and 144	145
2 and 72	74
3 and 48	51
4 and 36	40
6 and 24	30
$\mathbf{8}$ and $\mathbf{18}$	$\mathbf{26}$
9 and 16	25
12 and 12	24

To speed up the process, notice that the sums are automatically ordered from biggest to smallest -- so we don't necessarily have to create the whole table.

We could have started with some intermediate pair, say 6 and 24, and realized that since the sum is too big, we need the first factor to be bigger than 6.

Or, we could have noticed that sum of 12 and 12 is in the ballpark of 26, and worked our way up from the bottom of the table.

Case of Negative Terms

To deal with a negative value for b, we could use the same method as before, except that we would have to make both factors negative.

For example, since we know that 8 and 18 are factors of 144 that add to 26, we also know that -8 and -18 are factors of 144 that add to -26, so the expression $x^2 - 26x + 144$ factors to $(x-8)(x-18)$.

To deal with a negative value for c, we can think about the difference instead of the sum.

For example, to factor $x^2 + 32x - 144$, we can find which factor pair of 144 has a difference of 32, and put a negative on the smaller factor to make the sum. Since this pair is 4 and 36, the expression factors to $(x-4)(x+36)$.

Factor Pair	**Difference**
1 and 144	143
2 and 72	70
3 and 48	45
$\mathbf{4}$ and $\mathbf{36}$	$\mathbf{32}$
6 and 24	18
8 and 18	10
9 and 16	7
12 and 12	0

If b were negative as well -- say, if we wanted to factor $x^2 - 32x - 144$ -- then we could use the same process but put the negative on the bigger factor to make the sum negative. That is, we would put the negative on the 36 instead of the 4, and the resulting factored form would then be $(x+4)(x-36)$.

Case of a Common Factor

Sometimes, we can simplify quadratic expressions by factoring out something that ALL the terms have in common.

Original quadratic equation	$3x^2 - 15x + 18 = 0$
Factor a 3 out of all terms	$3(x^2 - 5x + 6) = 0$
Factor the quadratic expression	$3(x-2)(x-3) = 0$
Set each factor to 0	$x - 2 = 0 \text{ or } x - 3 = 0$
Solve	$x = 2 \text{ or } x = 3$

This makes it easy to factor quadratic expressions where c is 0 -- just factor out the variable!

Original quadratic equation	$x^2 + 7x = 0$
Factor an x out of all terms	$x(x+7) = 0$
Set each factor to 0	$x = 0 \text{ or } x + 7 = 0$
Solve	$x = 0 \text{ or } x = -7$

Case when the Leading Coefficient is Not One

Factoring out the variable works even when a is something other than 1.

Original quadratic equation	$2x^2 - 5x = 0$
Factor an x out of all terms	$x(2x-5) = 0$
Set each factor to 0	$x = 0 \text{ or } 2x - 5 = 0$
Solve	$x = 0 \text{ or } x = \frac{5}{2}$

But what about when a is something other than 1, and c is not zero?

There's a little trick that lets us reduce this to a factoring problem with a equal to 1. We multiply c by a, replace a with 1, factor the result, divide each constant in each factor by the original a, and move denominators onto our variables.

Original quadratic equation	$6x^2 + 11x + 3 = 0$
Multiply 3 by 6, and replace 6 with 1	$x^2 + 11x + 18 = 0$
Factor normally	$(x+9)(x+2) = 0$
Divide each constant in each factor by 6	$\left(x + \frac{9}{6}\right)\left(x + \frac{2}{6}\right) = 0$

Simplify	$\left(x+\frac{3}{2}\right)\left(x+\frac{1}{3}\right)=0$
Move denominators onto variables	$(2x+3)(3x+1)=0$
Set each factor to 0	$2x+3=0$ or $3x+1=0$
Solve	$x=-\frac{3}{2}$ or $x=-\frac{1}{3}$

We'll talk about why this trick works in the next chapter, when we cover the quadratic formula.

Case of No Middle Term

Lastly, what about when b is 0? Since the factors have to add to b, they must be negatives of each other. Since the factors have to multiply to c, and they are the same number (except one is negative), they must be the positive and negative square roots of c!

For example, x^2-4 factors to $(x+2)(x-2)$, and x^2-9 factors to $(x+3)(x-3)$.

This trick also works if a is not equal to 1 -- we just have to factor a out first.

Original quadratic equation	$3x^2-48=0$
Factor 3 out of all terms	$3(x^2-16)=0$

Factor the quadratic	$3(x+4)(x-4)=0$
Solve	$x=-4$ or $x=4$

Exercises

Factor the following quadratic equations. Then, use the factored form to find the solutions.

1) $x^2+7x+12=0$	2) $x^2+9x+14=0$
3) $x^2-7x=-10$	4) $x^2+18=9x$
5) $x^2+2x=8$	6) $21-4x=x^2$
7) $3x+10=x^2$	8) $x^2-5x=36$
9) $4x^2=-52x$	10) $-8x^2+64x=0$
11) $x^2-25=0$	12) $x^2-144=0$
13) $12x^2+11x=-2$	14) $10x^2=27x-5$
15) $5x=4-6x^2$	16) $21x^2-10=-29x$

2.3 Quadratic Formula

Some quadratic equations cannot be factored easily. For example, in the equation $x^2+3x+1=0$, we need to find two factors of 1 that add to 3. But the only integer factors of 1 are 1 and 1, and they definitely don't add to 3!

To solve these hard-to-factor quadratic equations, it's easiest to use the **quadratic formula** given below, which tells us explicitly how to compute the solutions of a quadratic equation $ax^2+bx+c=0$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Worked Example

Using the quadratic formula, we can compute the solutions to the equation $x^2+3x+1=0$.

Substitute $a=1$, $b=3$, and $c=1$ in quadratic formula	$x=\frac{-3\pm\sqrt{(3)^2-4(1)(1)}}{2(1)}$
Simplify	$x=\frac{-3\pm\sqrt{5}}{2}$
Separate the $\pm$ into two solutions (optional)	$x=\frac{-3+\sqrt{5}}{2}$ or $x=\frac{-3-\sqrt{5}}{2}$

These solutions look weird, but they're correct.

$$\begin{aligned} x^2+3x+1&=0 \\ \left(\frac{-3+\sqrt{5}}{2}\right)^2+3\left(\frac{-3+\sqrt{5}}{2}\right)+1&=0 \\ \frac{9-6\sqrt{5}+5}{4}+\frac{-9+3\sqrt{5}}{2}+1&=0 \\ \frac{7-3\sqrt{5}}{2}+\frac{-9+3\sqrt{5}}{2}+1&=0 \\ \frac{-2}{2}+1&=0 \\ 0&=0 \end{aligned}$$

$$\begin{aligned} x^2+3x+1&=0 \\ \left(\frac{-3-\sqrt{5}}{2}\right)^2+3\left(\frac{-3-\sqrt{5}}{2}\right)+1&=0 \\ \frac{9+6\sqrt{5}+5}{4}+\frac{-9-3\sqrt{5}}{2}+1&=0 \\ \frac{7+3\sqrt{5}}{2}+\frac{-9-3\sqrt{5}}{2}+1&=0 \\ \frac{-2}{2}+1&=0 \\ 0&=0 \end{aligned}$$

Reverse Derivation

To gain some faith in the quadratic formula, we can also rearrange it back into the original equation to see that it must have the same solutions as the original equation:

$$\begin{aligned} x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ 2ax&=-b\pm\sqrt{b^2-4ac} \\ 2ax+b&=\pm\sqrt{b^2-4ac} \\ (2ax+b)^2&=\left(\pm\sqrt{b^2-4ac}\right)^2 \\ 4a^2x^2+4abx+b^2&=b^2-4ac \\ 4a^2x^2+4abx+4ac&=0 \\ 4a\left(ax^2+bx+c\right)&=0 \\ ax^2+bx+c&=0 \end{aligned}$$

The Discriminant

Using the quadratic equation, we can see that some quadratic equations have 2 solutions (as usual), but other quadratic equations can have just 1 solution, or no solutions at all.

For example, using the quadratic equation to solve $x^2 + 2x + 1$, we find a single solution because the $\pm\sqrt{b^2 - 4ac}$ part comes out to ± 0.

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(1)}}{2(1)} = \frac{-2 \pm 0}{2} = -1$$

Similarly, using the quadratic equation to solve $x^2 + 2x + 2$, we find no solutions because the $\pm\sqrt{b^2 - 4ac}$ part comes out to $\pm\sqrt{\text{negative number}}$, and we can't take the square root of a negative number. (We'll ignore imaginary solutions and consider only real solutions for now.)

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \text{ no solution}$$

To see how many solutions a quadratic equation has, we need only consider the $b^2 - 4ac$ part of the quadratic formula, which is called the **discriminant**. If the discriminant is positive, then we have two solutions. If it is 0, then we have one solution. If it is negative, then we have no solution.

We can also use the quadratic formula to understand the trick for factoring when a is not equal to 1 -- which was to multiply c by a, replace a with 1, factor the result, divide each constant in each factor by the original a, and move denominators onto our variables.

From the quadratic formula, we know that the solutions of $ax^2+bx+c=0$ are given by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. When we multiply c by a and replace a with 1, we have the equation $x^2+bx+ac=0$, which has solutions $x=\frac{-b\pm\sqrt{b^2-4ac}}{2}$.

This means that if x is a solution of $x^2+bx+ac=0$, then ax is a solution of ax^2+bx+c.

Thus, if $x^2+bx+ac$ factors into $(x+m)(x+n)$, then ax^2+bx+c factors into $a\left(x+\frac{m}{a}\right)\left(x+\frac{n}{a}\right)$.

Exercises

Use the quadratic formula to solve the following quadratic equations.

1) $x^2+3x-7=0$

2) $4x^2-12x+9=0$

3) $-2x^2+4x+6=0$

4) $3x^2-x+5=0$

5) $-25x^2-20x=4$

6) $3x^2=2x+3$

7) $42x=9x^2+49$

8) $8x^2+5=3x$

9) $1+3x=5x^2$

10) $154x=121x^2+49$

2.4 Completing the Square

Completing the square is another method for solving quadratic equations. Although we can use the quadratic formula to solve any quadratic equation, completing the square helps us gain a better intuition for quadratic equations and understand where the quadratic formula comes from.

As we will see in the next chapter, completing the square will also help us rearrange quadratic equations into forms that are easy to graph.

Demonstration

The main idea behind completing the square is that every quadratic expression has a squared factor hidden inside of it.

Original equation	$x^2 + 2x - 5 = 0$
Add 5 to both sides	$x^2 + 2x = 5$
Add 1 to both sides	$x^2 + 2x + 1 = 6$
Factor	$(x+1)^2 = 6$
Take positive/negative root	$x + 1 = \pm\sqrt{6}$
Solve	$x = -1 \pm \sqrt{6}$

General Procedure

To find the squared factor, we just need to move the constant to the other side of the equation and add $\left(\frac{b}{2}\right)^2$ to both sides. Then, the quadratic expression will factor into $\left(x+\frac{b}{2}\right)^2$.

Original equation	$x^2+bx+c=0$
Move the constant to the other side	$x^2+bx=-c$
Add $\left(\frac{b}{2}\right)^2$ to both sides	$x^2+bx+\left(\frac{b}{2}\right)^2=\left(\frac{b}{2}\right)^2-c$
Factor	$\left(x+\frac{b}{2}\right)^2=\left(\frac{b}{2}\right)^2-c$
Take positive/negative root	$x+\frac{b}{2}=\pm\sqrt{\left(\frac{b}{2}\right)^2-c}$
Solve	$x=-\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-c}$

Hey, the solution is the same as the quadratic equation with $a = 1$!

$$x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$$
$$x = -\frac{b}{2} \pm \sqrt{\frac{b^2 - 4c}{4}}$$
$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4(1)c}}{2(1)}$$

Case when the Leading Coefficient is Not One

To complete the square with a not equal to 1, we can simply divide by a to create an equivalent equation where a IS equal to 1.

Original equation	$3x^2 - 18x - 1 = 0$
Divide by 3	$x^2 - 6x - \frac{1}{3} = 0$
Add $\frac{1}{3}$ to both sides	$x^2 - 6x = \frac{1}{3}$
Add $\left(\frac{-6}{2}\right)^2 = 9$ to both sides	$x^2 - 6x + 9 = \frac{28}{3}$
Factor	$(x-3)^2 = \frac{28}{3}$
Take positive/negative root and solve	$x = 3 \pm \sqrt{\frac{28}{3}}$

By completing the square on the general form $ax^2+bx+c=0$, we arrive at the quadratic equation:

Original equation	$ax^2+bx+c=0$
Divide by a	$x^2+\frac{b}{a}x+\frac{c}{a}=0$
Move the constant to the other side	$x^2+\frac{b}{a}x=-\frac{c}{a}$
Add $\left(\frac{b/a}{2}\right)^2=\left(\frac{b}{2a}\right)^2$ to both sides	$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$
Factor	$\left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$
Take positive/negative root	$x+\frac{b}{2a}=\pm\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}$
Solve	$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}$
Simplify	$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Exercises

Solve the following quadratic equations by completing the square. If there are two solutions, leave your answer in the form $\text{number} \pm \sqrt{\text{other number}}$.

1) $x^2 + 3x - 1 = 0$

2) $x^2 - x - 2 = 0$

3) $-x^2 + 2x + 3 = 0$

4) $-x^2 - 4x + 7 = 0$

5) $x^2 = 1 - 7x$

6) $3x = x^2 + 5$

7) $3x^2 - 3 = 2x$

8) $1 = 5x^2 + 3x$

9) $2x - 2x^2 = 5$

10) $3 - 7x^2 = x$

2.5 Vertex Form

To easily graph a quadratic equation, we can convert it to **vertex form**:

$$y = a(x-h)^2 + k$$

In vertex form, we can tell the coordinates of the vertex of the parabola just by looking at the equation: the vertex is at (h, k). We can also tell which way the parabola opens, by checking whether a is positive (opens up) or negative (opens down).

Equation	**Vertex**	**Opens**
$y = (x-2)^2 + 1$	$(2, 1)$	up
$y = -2(x-5)^2 - 3$	$(5, -3)$	down
$y = 7(x-1)^2 + \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$	up
$y = -\frac{1}{3}\left(x + \frac{2}{7}\right)^2 - \frac{4}{5}$	$\left(-\frac{2}{7}, -\frac{4}{5}\right)$	down

Converting to Vertex Form

To convert a quadratic equation into vertex form, we can complete the square.

Original equation	$y = 2x^2 - 4x - 5$
Divide by 2	$\frac{y}{2} = x^2 - 2x - \frac{5}{2}$
Move the constant to the other side	$\frac{y}{2} + \frac{5}{2} = x^2 - 2x$
Add $\left(\frac{-2}{2}\right)^2 = 1$ to both sides	$\frac{y}{2} + \frac{7}{2} = x^2 - 2x + 1$
Factor	$\frac{y}{2} + \frac{7}{2} = (x-1)^2$
Multiply by 2	$y + 7 = 2(x-1)^2$
Subtract 7	$y = 2(x-1)^2 - 7$

Exercises

Write the equation in vertex form $y = a(x-h)^2 + k$. Then, find the coordinates of the vertex and tell which way the parabola opens.

1) $y = x^2 + 2x + 3$

2) $y = x^2 - 6x + 4$

3) $y = 2x^2 + 20x - 5$

4) $y = -3x^2 + 6x + 1$

5) $y = 2x^2 - 2x - 1$

6) $y = -\frac{1}{3}x^2 - 2x - 10$

7) $y = \frac{3}{4}x^2 - 2x + 3$

8) $y = -\frac{2}{3}x^2 - 8x - 17$

2.6 Quadratic Systems

Systems of quadratic equations can be solved via substitution. After substituting, the resulting equation can itself be reduced down to a quadratic equation and solved by techniques covered in this chapter.

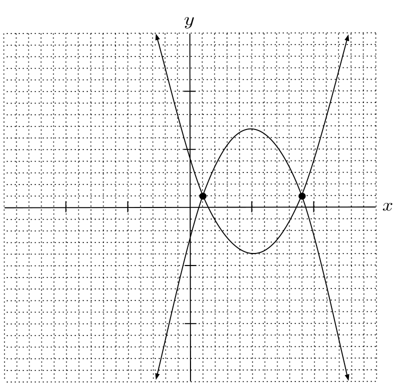
Original system	$\begin{cases} y = 2x^2 - x \\ y = x^2 - 3x + 3 \end{cases}$
Substitute for y	$2x^2 - x = x^2 - 3x + 3$
Convert to standard form	$x^2 + 2x - 3 = 0$
Solve for x	$x = -3 \text{ or } x = 1$
Evaluate y	$y = 2(-3)^2 - (-3) = 21$ or $y = 2(1)^2 - 1 = 1$
Solution	$(-3, 21) \text{ or } (1, 1)$

Note that when evaluating y, it doesn't matter which equation we use from the original equation. In the example above, we used the first equation because it was easier to compute, but using the second equation leads us to the same solutions.

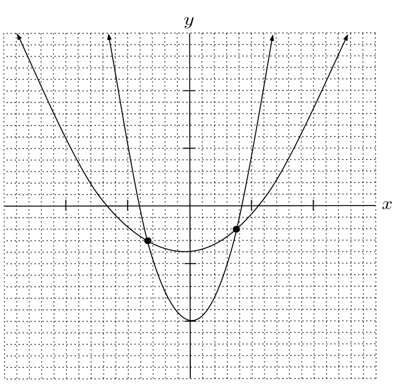
$$y = (-3)^2 - 3(-3) + 3 = 21$$
$$\text{or}$$
$$y = (1)^2 - 3(1) + 3 = 1$$

Number of Solutions

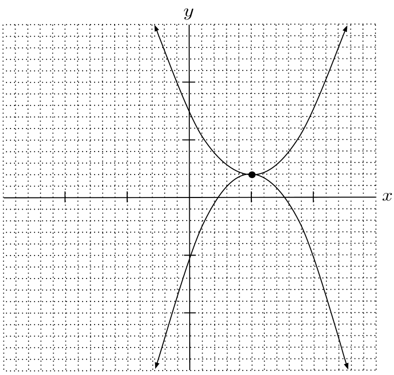
There can be 2, 1, or 0 points of intersection, depending on the arrangement of the parabolas.



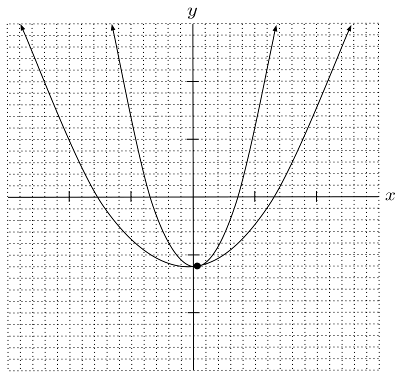
2 solutions



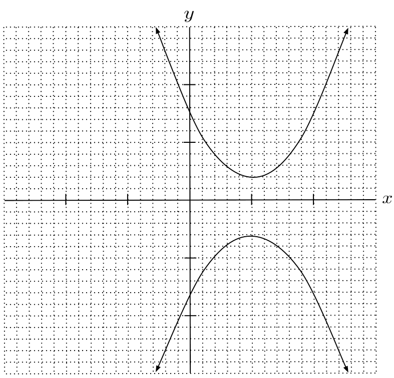
2 solutions



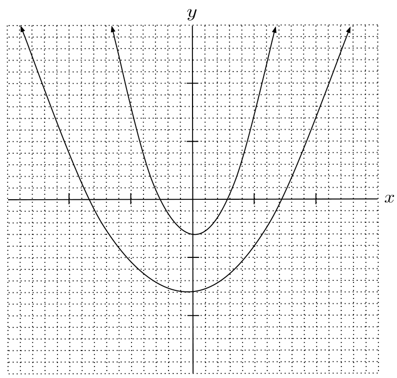
1 solution



1 solution



no solution



no solution

Just like in linear equations, if the result reduces down to a true statement, then there are infinitely many solutions because both equations in the system actually represent the same parabola.

Original system	$\begin{cases} y = x^2 + 2x - 3 \\ 2y = 2x^2 + 4x - 6 \end{cases}$
Substitute for y	$2(x^2 + 2x - 3) = 2x^2 + 4x - 6$
Simplify	$0 = 0$
Solution	all points on $y = x^2 + 2x - 3$

On the other hand, if the result reduces down to a false statement, then there are no solutions because the parabolas never intersect.

Original system	$\begin{cases} y = x^2 + 1 \\ y = x^2 - 1 \end{cases}$
Substitute for y	$x^2 + 1 = x^2 - 1$
Simplify	$1 = -1$
Solution	no solution

Exercises

Solve the following systems of quadratic equations.

1) $\begin{cases} y = x^2 + 3x + 6 \\ y = -x^2 + 13x - 6 \end{cases}$

2) $\begin{cases} y = 2x^2 + 6x + 3 \\ y = 2x^2 + 5x - 2 \end{cases}$

3) $\begin{cases} y = 3x^2 + 7x + 9 \\ y = 2x^2 + 7x + 10 \end{cases}$

4) $\begin{cases} y = x^2 + 3x + 2 \\ y = -2x^2 + x - 5 \end{cases}$

5) $\begin{cases} y = x^2 + 8x - 2 \\ y = x^2 + 5x \end{cases}$

6) $\begin{cases} y = -x^2 - 7x - 10 \\ y = -2x^2 - 14x - 20 \end{cases}$

7) $\begin{cases} y = 5x^2 - x + 7 \\ y = x^2 + 2x + 1 \end{cases}$

8) $\begin{cases} y = x^2 - 10x + 10 \\ y = -x^2 + x - 5 \end{cases}$

Chapter 3
Inequalities

3.1 Linear Inequalities in the Number Line

An **inequality** is similar to an equation, but instead of saying two quantities are equal, it says that one quantity is greater than or less than another.

For example, since 1 is greater than 0, we write $1 > 0$. Likewise, since 0 is less than 1, we write $0 < 1$.

If we write $x > 0$, then we mean that x can be 1, 2, π, 0.000001, or any other positive number. If we write $x < 0$, then we mean that x can be -1, -2, $-\pi$, -0.000001, or any other negative number.

"Or Equal To" Inequalities

We can also write $x \geq 0$ to mean that x is greater than or equal to 0.

In $x > 0$, the number 0 is not a valid solution for x because 0 is not greater than 0, but in $x \geq 0$, the number 0 is a valid solution because 0 is greater than *or equal to* 0.

Likewise, we can write $x \leq 0$ to mean that x is less than or equal to 0.

Solving Inequalities

Inequalities can be solved much like equations: we can perform algebraic manipulations to both sides of the equation until we isolate the variable.

Original inequality	$4x - 2 > 2x + 8$
Add 2 to both sides	$4x > 2x + 10$
Subtract $2x$ from both sides	$2x > 10$
Divide both sides by 2	$x > 5$

If we substitute any number that is greater than 5, it will satisfy the original inequality.

For example, if we substitute $x = 6$, then the original inequality becomes $22 > 20$, which is true. Likewise, if we substitute $x = 5.001$, then the original inequality becomes $18.004 > 18.002$, which is true.

On the other hand, if we substitute any number that is 5 or less, it will not satisfy the original inequality.

For example, if we substitute $x = 5$, then the original inequality becomes $18 > 18$, which is not true. Likewise, if we substitute $x = 4$, then the original inequality becomes $14 > 16$, which is not true.

Flipping the Inequality

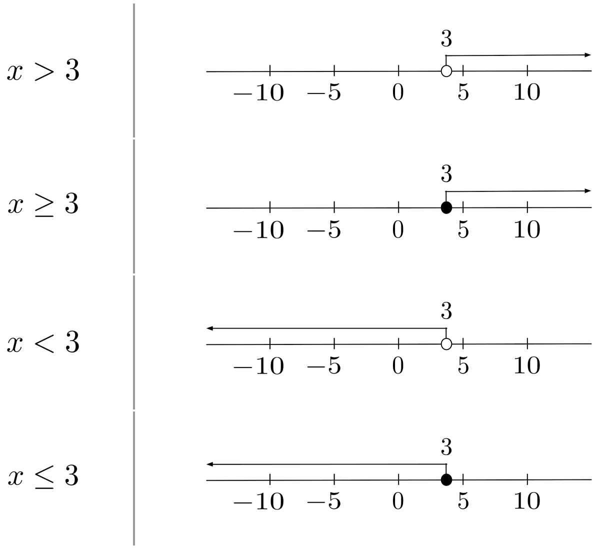
In manipulating inequalities, there is just one catch: **whenever we multiply or divide by a negative number, we have to flip the inequality.**

Original inequality	$x+1>5x-7$
Subtract 1 from both sides	$x>5x-8$
Subtract $5x$ from both sides	$-4x>-8$
Divide both sides by -4 and flip the inequality	$x<2$

To understand why we need to flip the inequality whenever we multiply or divide by a negative sign, consider the example $1<2$. If we multiply or divide by -1, we reach $-1<-2$, which is not true. In order to keep the inequality true, we have to flip the inequality sign: $-1>-2$.

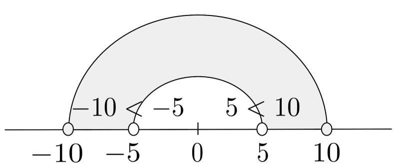
Plotting Inequalities

To visualize inequalities, we can plot them on a number line. An open (unfilled) circle around a point means that the point itself is NOT a solution, while a closed (filled) circle around a point means that the point itself is a solution.



The number line can help us understand why we have to flip the inequality sign whenever we multiply or divide by a negative number.

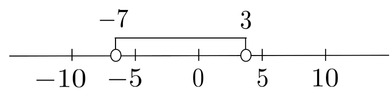
Starting with $5 < 10$, we know that 10 is the bigger number that is further from 0. When we multiply or divide, 10 is still going to be further from 0 than 5 is -- but if we multiply or divide by a negative number, then 10 will be further from 0 in the negative direction, which means it will actually be the lesser number.



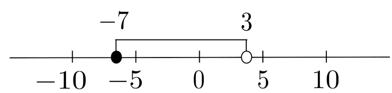
Interval Notation

The number line is a great intuitive aid, but it takes a while to draw. To simultaneously leverage the benefit of number line intuition and avoid the headache of drawing actual number lines, it is common to use **interval notation**, which represents number line segments using parentheses for open circles and brackets for closed circles.

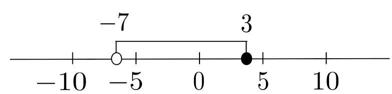
$$-7 < x < 3$$
$$(-7, 3)$$



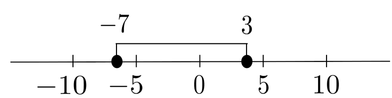
$$-7 \le x < 3$$
$$[-7, 3)$$



$$-7 < x \le 3$$
$$(-7, 3]$$

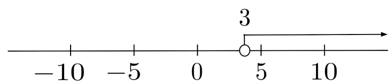


$$-7 \le x \le 3$$
$$[-7, 3]$$

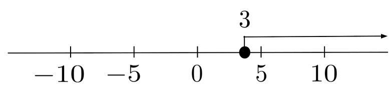


To indicate that a segment continues forever, we imagine it having an open circle at positive or negative infinity.

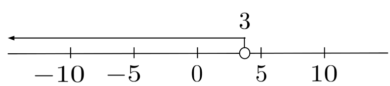
$x > 3$
$(3, \infty)$



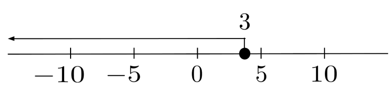
$x \geq 3$
$[3, \infty)$



$x < 3$
$(-\infty, 3)$



$x \leq 3$
$(-\infty, 3]$



Exercises

Solve the following inequalities, writing the solutions in interval notation.

1) $4x - 1 \leq 3x + 1$

2) $14 > 4 - 2x$

3) $-2x < 6x + 24$

4) $10 - 5x > 3 - 4x$

5) $10 - 7x \leq 3x - 5$

6) $22x + 2 \leq 16x$

7) $-4x - 7 < 2 - 5x$

8) $-3x < 4(x+3)$

9) $9x \geq 11(x-2) - 5x$

10) $3 - 4x > 4(1-2x) + 9x - 5$

3.2 Linear Inequalities in the Plane

When a linear equation has one variable, the solution covers a section of the number line: if our solution is $x > \text{some number}$, then the solution covers the section of the number line that lies right of that number; if our solution is $x < \text{some number}$, then the solution covers the section of the number line that lies left of that number.

If equality is allowed (i.e. $\geq$ or $\leq$), then we use a closed circle to indicate that the circled number is itself a solution; otherwise, if equality is not allowed (i.e. $>$ or $<$), then we use an open circle.

Similarly, when a linear equation has two variables, the solution covers a section of the coordinate plane. If our solution is $y > mx + b$, then the solution covers the section of the coordinate plane that lies above the line $y = mx + b$, whereas if our solution is $y < mx + b$, then the solution covers the section of the coordinate plane that lies below the line $y = mx + b$.

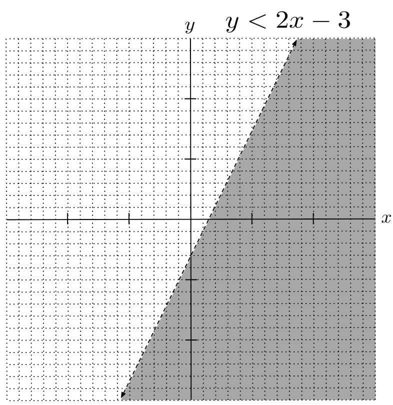
If equality is allowed (i.e. $\geq$ or $\leq$), then we use a solid line to indicate that points on the line itself are solutions. Otherwise, if equality is not allowed (i.e. $>$ or $<$), then we use a dotted line.

Worked Example

To illustrate, let's solve and graph a two-variable linear inequality.

Original inequality	$3y - x > 5(y - x) + 6$
Simplify	$3y - x > 5y - 5x + 6$
Subtract $5y$ from both sides	$-2y - x > -5x + 6$
Add x from both sides	$-2y > -4x + 6$
Divide by -2	$y < 2x - 3$

Since equality is not allowed in the solution, we draw a dotted line. Since the solution consists of values of y LESS THAN those on the line, we shade under the line.



We can check that any point in the shaded region is a solution: for example, substituting $(5,-10)$ into the original inequality yields $3(-10)-5>5(-10-5)+6$, which simplifies to $-35>-69$, which is true.

Likewise, we can check that any point NOT in the shaded region is NOT a solution: for example, substituting $(0,0)$ into the original inequality yields $3(0)-0>5(-0-0)+6$, which simplifies to $0>6$, which is not true

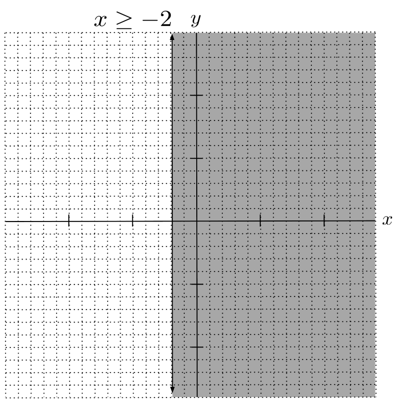
Any point on the line itself will not be a solution, but would be a solution if equality were allowed: for example, substituting the y-intercept $(0,-3)$ into the original inequality yields $3(-3)-0>5(-3-0)+6$, which simplifies to $-9>-9$, which is not a solution but would be a solution if equality were allowed (i.e. $9\geq-9$).

Case when a Variable Vanishes

If y vanishes while solving the equation, then the boundary line will be vertical. In this case, we shade left or right of the line depending on whether the solution tells us that x is less than some number, or greater than some number.

Original inequality	$-(y+2) \leq x - y$
Simplify	$-y - 2 \leq x - y$
Add y to both sides	$-2 \leq x$
Move x to left side	$x \geq -2$

Since equality is allowed in the solution, we draw a solid line. Since the solution consists of values of x GREATER THAN those on the line, we shade on the right towards higher values of x.



Exercises

Graph the solutions to the inequalities below.

1) $y < -3x + 8$

2) $y \leq \frac{1}{4}x - 7$

3) $y \geq 5(x + 4) - 2$

4) $y < -\frac{3}{8}(x - 2) + 3$

5) $2x + 14 \geq 11$

6) $3x - 4y < 12$

7) $-5y - 7x > 8x - 25$

8) $2(x + 3y + 2) < x + 6y + 2$

9) $\frac{1}{4}(x + 2y) < \frac{1}{8}(16 + 2x)$

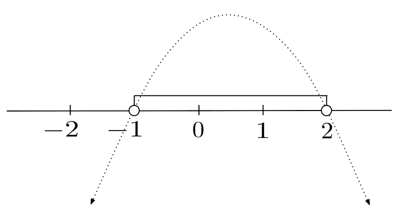
10) $x > \frac{1}{2}(8y + 30)$

3.3 Quadratic Inequalities

Quadratic inequalities are best visualized in the plane. For example, to solve a quadratic inequality $-x^2 + x + 2 > 0$, we can find the values of x where the parabola $y = -x^2 + x + 2$ is positive.

Since $y = -x^2 + x + 2$ is a downward parabola, the solution consists of the values of x in its midsection which arches over the x-axis. That is, the solution consists of all x-values between the solutions to $-x^2 + x + 2 = 0$.

This quadratic equation factors to $-(x-2)(x+1) = 0$, so the parabola's midsection is given by $-1 < x < 2$, or $(-1, 2)$ in interval notation.

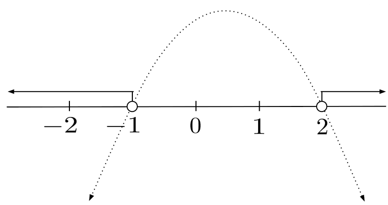


Case when the Solution is a Union

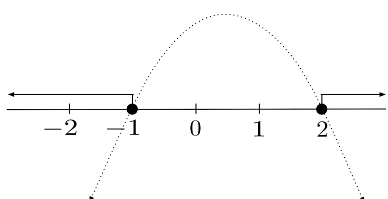
On the other hand, if we want to solve $-x^2 + x + 2 < 0$, then we need to find the values of x where the parabola $y = -x^2 + x + 2$ is negative.

This time, the solution consists of all the values of x in the arms of the parabola which extend under the x-axis. That is, the solution consists of all x-values less than the leftmost solution or greater than the rightmost solution to $-x^2 + x + 2 = 0$.

The solution of the inequality is then given by $x < -1 \text{ or } x > 2$, which is $(-\infty, -1) \cup (2, \infty)$ in interval notation. (The $\cup$ symbol is called a **union**, and it allows us to include multiple segments in interval notation.)



To solve $-x^2 + x + 2 \leq 0$, we just need to propagate the allowance of equality to our final answer. Thus, the solution is $x \leq -1 \text{ or } x \geq 2$, which is $(-\infty, -1] \cup [2, \infty)$ in interval notation.



Case when the Parabola is Never Zero

When a quadratic inequality involves a parabola that is never zero, there is either no solution or the solution is all real numbers.

For example, the parabola $y = x^2 + 5$ has only positive y-values, so $x^2 + 5 < 0$ has no solution and $x^2 + 5 > 0$ is solved by all real numbers.

In interval notation, we express all real numbers as the full number line $(-\infty, \infty)$, and we express no solution as $\emptyset$. (The $\emptyset$ symbol is called the **empty set**, and it represents an interval which doesn't contain any numbers.)

Exercises

Solve the following inequalities, writing the solutions in interval notation.

1) $x^2 - 4 \geq 0$

2) $x^2 + 1 > 0$

3) $-3x^2 + 27 < 0$

4) $x^2 - 9x + 14 < 0$

5) $x^2 + x + 1 \leq 0$

6) $2x^2 \geq 7x + 15$

7) $x + 1 < 6x^2$

8) $-x^2 - 7x + 30 \geq x^2 + 7x - 30$

9) $10x(x-3) \leq 7+3x$

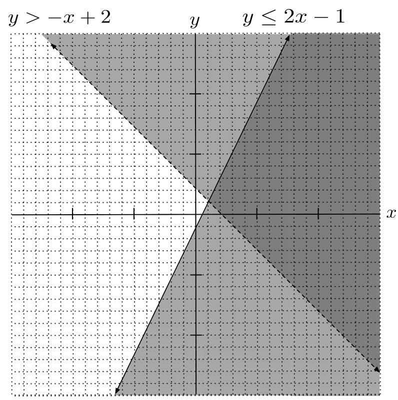
10) $(3x+5)^2 \geq 3(x^2-1)+7(x-11)$

3.4 Systems of Inequalities

To solve a **system of inequalities**, we need to solve each individual inequality and find where all their solutions overlap. For example, to solve the system

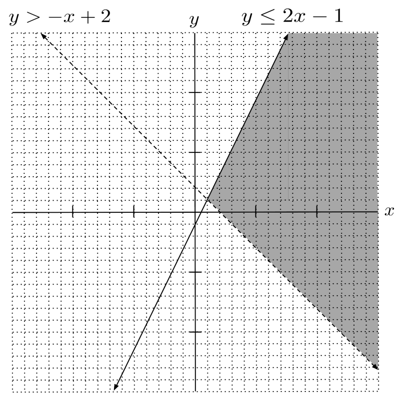
$$\begin{cases} y > -x + 2 \\ y \leq 2x - 1 \end{cases}$$

we first graph each individual inequality and darken where the shading overlaps.



The solution to the system consists of points that satisfy BOTH individual inequalities, so the solution is just the overlap of the two shadings, which appears as the most darkened part of the graph.

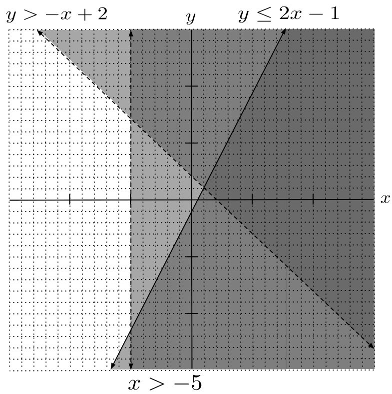
To display the solution to the system, we erase any other shading and shade only the overlap.

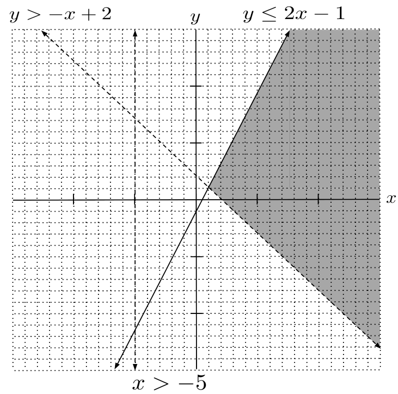


Including Another Inequality

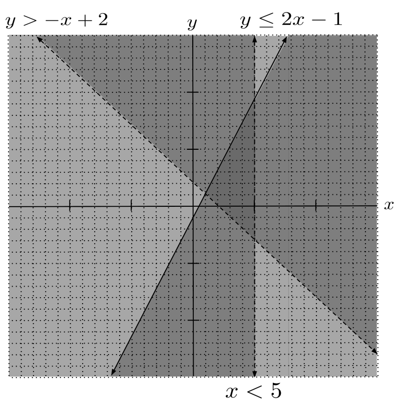
If we include another inequality in the system, then the solution region will either stay the same or shrink.

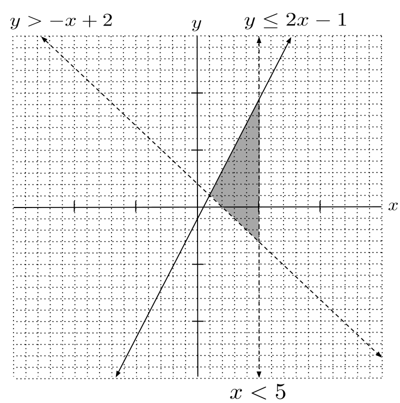
For example, if we include $x > -5$, then the solution region will stay the same because it is fully contained in the shading of $x > -5$
.





However, if we include $x < 5$, then the solution region will shrink because only part of it is contained in the shading of $x < 5$.

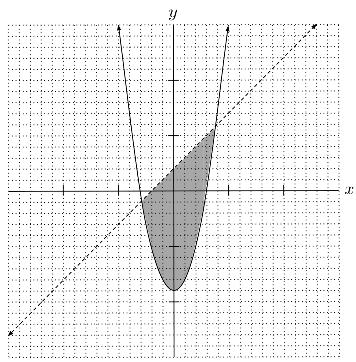




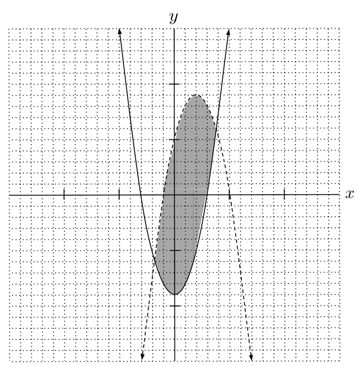
Quadratic Inequalities

Even with quadratic inequalities, the method is the same: the solution is the overlap of the shading of the component inequalities. Some examples are shown below.

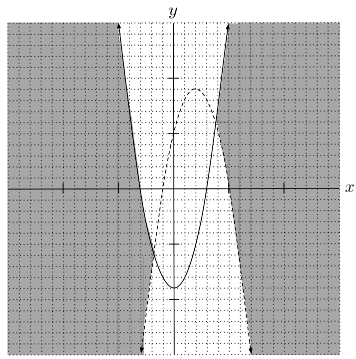
$$\begin{cases} y \geq x^2 - 9 \\ y < x + 2 \end{cases}$$



$$\begin{cases} y \geq x^2 - 9 \\ y < -x^2 + 4x + 5 \end{cases}$$



$$\begin{cases} y < x^2 - 9 \\ y \geq -x^2 + 4x + 5 \end{cases}$$



Exercises

Graph the solutions to the systems below.

1) $\begin{cases} y \geq 3x - 7 \\ y < -2x + 1 \end{cases}$

2) $\begin{cases} y < \frac{1}{3}x - 3 \\ y < -5x + 5 \end{cases}$

3) $\begin{cases} y \geq 2x + 1 \\ y \leq 8 \end{cases}$

4) $\begin{cases} x > -5 \\ y > x - 3 \\ y < 0 \end{cases}$

5) $\begin{cases} x + y < 1 \\ 3y < x - 3 \\ 2(x - 3) \geq x \end{cases}$

6) $\begin{cases} y \geq x^2 - 4 \\ y < 10 - x \\ y - x > 0 \end{cases}$

7) $\begin{cases} 2y + 5x^2 \leq 20 \\ y + 9 > x^2 \\ 2x - y \geq 4 \end{cases}$

8) $\begin{cases} y < x^2 \\ y + x^2 > 5 \\ y > 2(x-5) \\ x > 1 \end{cases}$

Part 4
Polynomials

4.1 Standard Form and End Behavior

Polynomials include linear expressions and quadratic expressions, as well as expressions adding multiples of higher exponents of the variable.

For example, these are polynomials:

$$x^5 - 4x^2 + 1 \qquad 3x^4 + (x+1)^3 \qquad x^6 + (x^2-1)^3 + x + 1$$

On the other hand, these are not polynomials:

$$x^2 + \sqrt{x} + 1 \qquad \sin(x) + x^3 - 2 \qquad |x| + x^5 + x^3 - 1$$

Standard Form

Polynomials are usually written in **standard form**, in which all terms are fully expanded and variable exponents are arranged from greatest to least.

Original polynomial	$x(5+2x^2)(x-1) + 3(4+x^3)$
Simplify	$x(5x - 5 + 2x^3 - 2x^2) + 12 + 3x^3$ $5x^2 - 5x + 2x^4 - 2x^3 + 12 + 3x^3$

Combine like terms	$5x^2 - 5x + 2x^4 + x^3 + 12$
Arrange exponents from greatest to least	$2x^4 + x^3 + 5x^2 - 5x + 12$

End Behavior

The end behavior of a polynomial refers to how it behaves when we substitute extremely large positive or negative values for x.

If the polynomial evaluates to a very large positive number, we say it approaches infinity. Otherwise, if the polynomial evaluates to a very large negative number, we say it approaches negative infinity.

For example, consider the polynomial $p(x) = -2x^3 + x^2 + 5x - 3$. When we substitute a large positive number, such as $x = 100$, the output is a large negative number.

$$\begin{aligned} p(100) &= -2(100)^3 + (100)^2 + 5(100) - 3 \\ &= -2(1000000) + 10000 + 500 - 3 \\ &= -2000000 + 10000 + 500 - 3 \\ &= -1989503 \end{aligned}$$

When we substitute a large negative number, such as $x = -100$, the output is a large positive number.

$$\begin{aligned} p(-100) &= -2(-100)^3 + (-100)^2 + 5(-100) - 3 \\ &= -2(-1000000) + 10000 - 500 - 3 \\ &= 2000000 + 10000 - 500 - 3 \\ &= 2009497 \end{aligned}$$

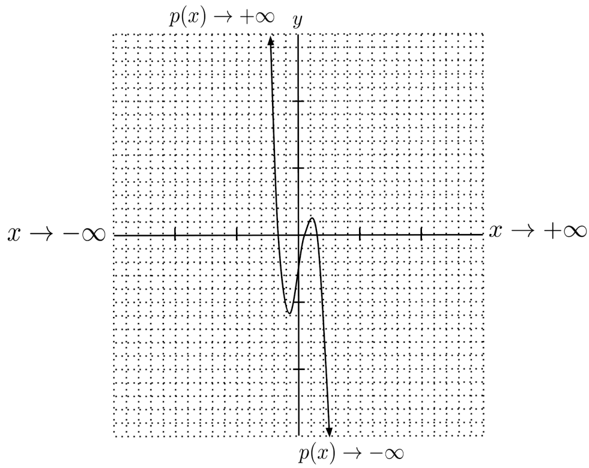
Putting this together, we say that $p(x)$ goes to negative infinity as x goes to positive infinity, and $p(x)$ goes to positive infinity as x goes to negative infinity.

We can write this symbolically: $p(x) \to -\infty$ as $x \to +\infty$, and $p(x) \to +\infty$ as $x \to -\infty$. This is the end behavior of the polynomial $p(x)$.

Graphical Interpretation

Graphically, end behavior tells us whether the polynomial curves up or down as we travel away from the origin in the right or left direction.

Since $p(x) \to -\infty$ as $x \to +\infty$, we know that the polynomial curves down as we travel to the right, and since $p(x) \to +\infty$ as $x \to -\infty$, we know that the polynomial curves up as we travel to the left.



Shortcuts

Do you notice any patterns or shortcuts? It's possible to determine the end behavior of a polynomial without evaluating the full polynomial.

The term with the highest exponent controls the end behavior, because it makes the greatest contribution to the result. All the other terms make much smaller contributions -- they're peanuts in comparison to the highest-exponent term.

$$\begin{aligned} p(100) &\approx -2(100)^3 = -2000000 \\ p(-100) &\approx -2(-100)^3 = 2000000 \end{aligned}$$

But we can do even better -- we don't actually have to evaluate anything at all! Within the term having the highest exponent, we just need to look at the exponent and sign of the coefficient. If the exponent is even, then the result after exponentiation will always be positive. Consequently, the term will evaluate to have the same sign as its coefficient.

For example, to find the end behavior of the polynomial $p(x) = 2x^2 - 3x + 4$, we just need to look at the $2x^2$ term. Since the exponent is even, x^2 will always be positive -- if we substitute $x = 100$, then $x^2 = 10000$, and if we substitute $x = -100$, then $x^2 = 10000$ again. The coefficient 2 is also positive, so $2x^2$ is always a positive times a positive, which makes a positive. As a result, we have $p(x) \to +\infty$ as $x \to +\infty$ and $p(x) \to +\infty$ as $x \to -\infty$.

Likewise, to find the end behavior of the polynomial $p(x) = -5x^4 + 7x^3 - x - 2$, we just need to look at the $-5x^2$ term. Since the exponent is even, x^4 will always be positive -- if we substitute $x = 100$, then $x^4 = 100000000$, and if we substitute $x = -100$, then $x^4 = 100000000$ again. But the coefficient -5 is negative, so $-5x^4$ is always a negative times a positive, which makes a negative. As a result, we have $p(x) \to -\infty$ as $x \to +\infty$ and $p(x) \to -\infty$ as $x \to -\infty$.

Examples with an Odd Exponent

On the other hand, if the exponent is odd, then the result after exponentiation will always have the same sign as the input x. Consequently, the term will evaluate to be positive if the coefficient and the input x have the same sign, and negative if they have opposite signs.

For example, to find the end behavior of the polynomial $p(x) = 4x^3 - 5x^2 - 2x + 1$, we just need to look at the $4x^3$ term. Since the exponent is odd, exponentiation will not change the sign -- if we substitute $x = 100$, then $x^3 = 1000000$, and if we substitute $x = -100$, then $x^3 = -1000000$. The coefficient 4 is positive, and multiplying by a positive doesn't change the sign either. As a result, we have $p(x) \to +\infty$ as $x \to +\infty$ and $p(x) \to -\infty$ as $x \to -\infty$
.
Likewise, to find the end behavior of the polynomial $p(x) = -3x^5 + 7x^4 + 3x^2 - 10x$, we just need to look at the $-3x^5$ term. Since the exponent is odd, exponentiation will not change the sign -- if we substitute $x = 100$, then $x^5 = 10000000000$, and if we substitute $x = -100$, then $x^5 = -10000000000$. But the coefficient -3 is negative, and multiplying by a negative changes the sign -- if $x^5 = 10000000000$, then $-3x^5 = -30000000000$, and if $x^5 = -10000000000$, then $-3x^5 = 30000000000$. As a result, we have $p(x) \to -\infty$ as $x \to +\infty$ and $p(x) \to +\infty$ as $x \to -\infty$.

Exercises

Convert the following polynomials to standard form. Then, write their end behavior symbolically: $p(x) \to$ ___ as $x \to +\infty$, and $p(x) \to$ ___ as $x \to -\infty$.

1) $p(x) = 3x^4 - 7x + 8x^3$

2) $p(x) = 1 - 2x + x^3$

3) $p(x) = 2 + 3x^3 - x^6 - 9x^4$

4) $p(x) = -10x^9 - x^3 + 5x^4 - x^2$

5) $p(x) = 6x^3 - 4x^5 + 1 - 2x$

6) $p(x) = 1 + 4x^3 - 5x + x^4$

7) $p(x) = 8x - 1 + x^4 - 5x^6 - 6x^5$

8) $p(x) = -x^4 + x^{11} - 3x^8 + 2$

9) $p(x) = (x^2 + 1)(x + 1)$

10) $p(x) = -(x - 1)(x + 1)(x - 3)$

11) $p(x) = (2x^5 + 1)(x^3 - 2)$

12) $p(x) = x^2(3 - 2x^4)(x + 2)$

4.2 Zeros

The **zeros** of a polynomial are the inputs that cause it to evaluate to zero.

For example, a zero of the polynomial $x^3 - 2x^2 - 5x + 6$ is $x = 1$ because $(1)^3 - 2(1)^2 - 5(1) + 6 = 0$. Another zero is $x = -2$ because $(-2)^3 - 2(-2)^2 - 5(-2) + 6 = 0$. Can you find the rest?

Finding Zeros by Factoring

One trick for finding the zeros of polynomials is to write the polynomial in factored form.

Since we know that $x = 1$ and $x = -2$ are zeros of the polynomial, we know the polynomial has to have factors $x - 1$ and $x + 2$. If we multiply these factors together, we get a polynomial whose highest-exponent term is x^2.

But our original polynomial has a highest-exponent term of x^3, so we need to multiply by one more factor. Consequently, the factored polynomial will take the form $(x-1)(x+2)(x-a)$ for some other zero $x = a$.

$$x^3 - 2x^2 - 5x + 6 = (x-1)(x+2)(x-a)$$

Let's multiply out the factors and group like terms into the form of the original polynomial.

$$\begin{aligned} x^3 - 2x^2 - 5x + 6 &= (x-1)(x+2)(x-a) \\ &= (x^2 + x - 2)(x-a) \\ &= x^3 + x^2 - ax^2 - 2x - ax + 2a \\ &= x^3 - (a-1)x^2 - (a+2)x + 2a \end{aligned}$$

From here, we can proceed in any of several different ways to discover that $a = 3$.

- The x^2 coefficient of the right-hand side is $-(a-1)$, and the x^2 coefficient of the left-hand side is -2, so we need $-(a-1) = -2$, which means $a = 3$.
- The x coefficient of the right-hand side is $-(a+2)$, and the x coefficient of the left-hand side is -5, so we need $-(a+2) = -5$, which means $a = 3$.
- The constant coefficient of the right-hand side is $2a$, and the constant coefficient of the left-hand side is 6, so we need $2a = 6$, which means $a = 3$.

Indeed, checking our answer, we find that substituting $x = 3$ makes the polynomial evaluate to 0.

$$(3)^3 - 2(3)^2 - 5(3) + 6 = 0$$

Fundamental Theorem of Algebra

Through this example, we've learned an important thing about the zeros of polynomials: **the number of zeros of a polynomial is no more than its degree**.

Each zero comes from a factor, and the degree of a polynomial limits the amount of factors it has, which in turn limits the amount of zeros it has. A third-degree polynomial can't have more than 3 factors, so it has at most 3 zeros. A tenth-degree polynomial can't have more than 10 factors, so it has at most 10 zeros.

Some polynomials look like they have fewer zeros than their degree -- for example, the polynomial $x^2 + 1$ doesn't appear to have any zeros, because there is no solution to $x^2 = -1$. But if we allow the use of the imaginary unit $i = \sqrt{-1}$, then it does have two zeros: $x = i$ and $x = -i$.

Likewise, the polynomial $x^2 + 2x + 1$ factors to $(x+1)^2$ and thus appears to have only one zero, $x = -1$. But since this factor is squared, we can think of counting the $x = -1$ zero twice, i.e. it has a **multiplicity** of two.

This is the **fundamental theorem of algebra:** the number of zeros of a polynomial is equal to its degree, provided we allow the use of the imaginary unit and count zeros according to their multiplicity.

Solving a Polynomial Equation

Finding zeros of polynomials is important because of its generality: every polynomial equation reduces to finding the zeros of some polynomial.

For example, consider the polynomial equation $x^3 + 5x^2 = 11x - x^3 - 4$, for which we can see that $x = 1$ is a solution because $1 + 5 = 11 - 1 - 4$. Subtracting $11x - x^3 - 4$ from both sides, we reach $2x^3 + 5x^2 - 11x + 4 = 0$. Now, the problem is to find the zeros of the polynomial $2x^3 + 5x^2 - 11x + 4$.

The polynomial has degree 3, so we are looking for 3 zeros, each of which corresponds to a factor of the polynomial. We know one of the zeros is $x = 1$, which corresponds to a factor $x - 1$, and we know the other two factors need to multiply to a quadratic $2x^2 + bx + c$.

By multiplying out $(x-1)(2x^2 + bx + c)$ and comparing coefficients to the original polynomial, we can solve for b and c. Then, we can solve the quadratic to find the remaining zeros.

$$\begin{aligned} 2x^3 + 5x^2 - 11x + 4 &= (x-1)(2x^2 + bx + c) \\ &= 2x^3 + bx^2 + cx - 2x^2 - bx - c \\ &= 2x^3 + (b-2)x^2 + (c-b)x - c \end{aligned}$$

Equating x^2 coefficients, we see that $5 = b - 2$, so $b = 7$. Finally, by equating the constants 4 and $-c$, we see that $c = -4$. The polynomial can then be written as

$$(x-1)(2x^2+7x-4).$$

Solving the quadratic $2x^2+7x-4=0$ leads us to the two remaining zeros: $x=-4$ and $x=\frac{1}{2}$.

We check to ensure that these zeros are indeed solutions of the original equation:

$$\begin{aligned}(-4)^3+5(-4)^2 &= 11(-4)-(-4)^3-4 \\ 16 &= 16\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{2}\right)^3+5\left(\frac{1}{2}\right)^2 &= 11\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^3-4 \\ \frac{11}{8} &= \frac{11}{8}\end{aligned}$$

Exercises

For each of the following polynomials, use the given zero(s) to find the remaining zero(s).

1) $p(x) = 2x^3 - x^2 - 2x + 1$
   given zeros: $1, -1$

2) $p(x) = 4x^3 - 8x^2 - 59x + 63$
   given zero: 1

3) $p(x) = 3x^4 - 22x^3 + 41x^2 + 2x - 24$
   given zeros: $1, 3$

4) $p(x) = 49x^5 - 133x^4 + 15x^3 + 145x^2 - 64x - 12$
   given zeros: $1, -1, 2$

For each of the following equations, use the given solution(s) to find the remaining solution(s).

5) $x^3 + 17x = 8x^2 + 10$
   given solutions: $1, 2$

6) $x(x^2 - 11) = 2(x^2 - 6)$
   given solution: 1

7) $4x^3 + 26x + 5x^4 = 63x^2 - 88$
given solutions: $-1, 2$

8) $x(x^4 + 144x - 47) = 2(3x^4 + 13x^3 + 105)$
given solutions: $-1, 2, 3$

4.3 Rational Roots and Synthetic Division

In the previous chapter, we learned how to find the remaining zeros of a polynomial if we are given some zeros to start with. But how do we get those initial zeros in the first place, if they're not given to us and aren't obvious from the equation?

Rational Roots Theorem

The **rational roots theorem** can help us find some initial zeros without blindly guessing. It states that for a polynomial with integer coefficients, any rational number (i.e. any integer or fraction) that is a root (i.e. zero) of the polynomial can be written as some factor of the constant coefficient, divided by some factor of the leading coefficient.

For example, if the polynomial $p(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$ has a rational root, then it is some positive or negative fraction having numerator 1 or 3 and denominator 1 or 2.

The possible roots are then $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$, or ± 3. We test each of them below.

$$\mathbf{p}\left(\tfrac{\mathbf{1}}{\mathbf{2}}\right) = \mathbf{0} \qquad p\left(-\tfrac{1}{2}\right) = \tfrac{11}{4} \qquad p(1) = -4 \qquad \mathbf{p(-1) = 0}$$

$$p\left(\tfrac{3}{2}\right) = -\tfrac{15}{4} \qquad p\left(-\tfrac{3}{2}\right) = -\tfrac{3}{2} \qquad p(3) = 120 \qquad p(-3) = 84$$

We see that $x = \frac{1}{2}$ and $x = -1$ are indeed zeros of the polynomial.

Therefore, the polynomial can be written as

$$\left(x - \tfrac{1}{2}\right)(x+1)(2x^2 + bx + c)$$

for some constants b and c, which we can find by expanding and matching up coefficients.

$$\begin{aligned} 2x^4 + x^3 - 7x^2 - 3x + 3 &= \left(x - \frac{1}{2}\right)(x+1)(2x^2 + bx + c) \\ &= 2x^4 + (1+b)x^3 + \left(c + \frac{b}{2} - 1\right)x^2 + \left(\frac{c}{2} - \frac{b}{2}\right)x - \frac{c}{2} \end{aligned}$$

We find that $b = 0$ and $c = -6$.

The remaining quadratic factor becomes $2x^2 - 6$, which has zeros $x = \pm\sqrt{3}$.

Thus, the zeros of the polynomial are $-\sqrt{3}$, -1, $\frac{1}{2}$, and $\sqrt{3}$.

Synthetic Division

To speed up the process of finding the zeros of a polynomial, we can use **synthetic division** to test possible zeros and update the polynomial's factored form and rational roots possibilities each time we find a new zero.

Given the polynomial $x^4 + 3x^3 - 5x^2 - 21x - 14$, the rational roots possibilities are ± 1, ± 2, ± 7, and ± 14.

To test whether, say, 2 is a zero, we can start by setting up a synthetic division template which includes 2 at the far left, followed by the coefficients of the polynomial (in the order that they appear in standard form).

$$\begin{array}{c|ccccc} 2 & 1 & 3 & -5 & -21 & -14 \\ & & & & & \\ \hline \end{array}$$

We put a 0 under the first coefficient (in this case, 1) and add down the column.

Then, we multiply the result by the leftmost number (in this case, 2) and put it under the next coefficient (in this case, 3).

We repeat the same process over and over until we finish the final column.

$$\begin{array}{c|ccccc} 2 & 1 & 3 & -5 & -21 & -14 \\ & 0 & 2 & 10 & 10 & -22 \\ \hline & 1 & 5 & 5 & -11 & -36 \end{array}$$

The bottom-right number is the remainder when we divide the polynomial by the factor corresponding to the zero being tested. Therefore, if the bottom-right number is 0, then the top-left number is indeed a zero of the polynomial, because its corresponding factor is indeed a factor of the polynomial.

In this case, though, the bottom-right number is not 0 but -36, so 2 is NOT a zero of the polynomial.

However, when we repeat synthetic division with -2, the bottom-right number comes out to 0 and we conclude that -2 is a zero of the polynomial.

$$\begin{array}{r|rrrrr} -2 & 1 & 3 & -5 & -21 & -14 \\ & 0 & -2 & -2 & 14 & 14 \\ \hline & 1 & 1 & -7 & -7 & 0 \end{array}$$

Then $x + 2$ is a factor of the polynomial, and the bottom row gives us the coefficients in the sub-polynomial that multiplies $x + 2$ to yield the original polynomial.

$$\begin{aligned} x^4 + 3x^3 - 5x^2 - 21x - 14 &= (x+2)(1x^3 + 1x^2 - 7x - 7) \\ &= (x+2)(x^3 + x^2 - 7x - 7) \end{aligned}$$

The next factor will come from $x^3 + x^2 - 7x - 7$, so the rational roots possibilities are just ± 1 and ± 7.

We use synthetic division to test whether $x = 1$ is a zero of $x^3 + x^2 - 7x - 7$.

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -7 & -7 \\ & 0 & 1 & 2 & -5 \\ \hline & 1 & 2 & -5 & -12 \end{array}$$

Since the bottom-right number is -12 rather than 0, we see that $x = 1$ is not a zero of $x^3 + x^2 - 7x - 7$. However, $x = -1$ is!

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -7 & -7 \\ & 0 & -1 & 0 & 7 \\ \hline & 1 & 0 & -7 & 0 \end{array}$$

Using the bottom row as coefficients, we update the factored form of our polynomial.

$$\begin{aligned} x^4 + 3x^3 - 5x^2 - 21x - 14 &= (x+2)(x^3 + x^2 - 7x - 7) \\ &= (x+2)(x+1)(1x^2 + 0x - 7) \\ &= (x+2)(x+1)(x^2 - 7) \end{aligned}$$

Now that we're down to a quadratic, we can solve it directly.

$$\begin{aligned} x^2 - 7 &= 0 \\ x^2 &= 7 \\ x &= \pm\sqrt{7} \end{aligned}$$

Thus, the zeros of the polynomial are -2, -1, $\sqrt{7}$, and $-\sqrt{7}$, and the factored form of the polynomial is

$$(x+2)(x+1)(x+\sqrt{7})(x-\sqrt{7}).$$

Final Remarks

In this example, the polynomial factored fully into linear factors. However, if the last factor were x^2+7, which does not have any zeros, we would leave it in quadratic form. The zeros of the polynomial would be just -2 and -1, and the fully factored form of the polynomial would be $(x+2)(x+1)(x^2+7)$.

One last thing about synthetic division: be sure to include ALL coefficients of the original polynomial in the top row of the synthetic division setup, even if they are 0. For example, the polynomial $3x^4+2x$ is really $3x^4+0x^3+0x^2+2x+0$, so the top row in the synthetic division setup should read $3 \quad 0 \quad 0 \quad 2 \quad 0$.

Exercises

For each polynomial, find all the zeros and write the polynomial in factored form.

1) $3x^3 + 18x^2 + 33x + 18$

2) $2x^3 - 5x^2 - 4x + 3$

3) $x^4 - 4x^3 + 3x^2 + 4x - 4$

4) $x^4 - 3x^3 + 5x^2 - 9x + 6$

5) $x^4 - 2x^3 - x^2 + 4x - 2$

6) $2x^4 + 7x^3 + 6x^2 - x - 2$

7) $2x^4 - 8x^3 + 10x^2 - 16x + 12$

8) $21x^5 + 16x^4 - 74x^3 - 61x^2 - 40x - 12$

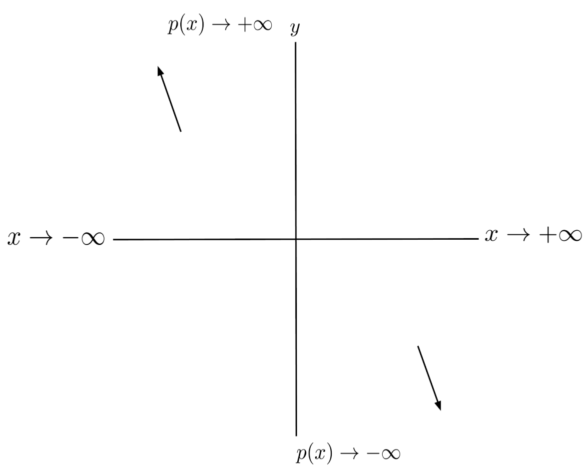
4.4 Sketching Graphs

In the previous chapters, we learned how to find end behavior, zeros, and factored forms of polynomials. In this chapter, we will put all this information together to sketch graphs of polynomials.

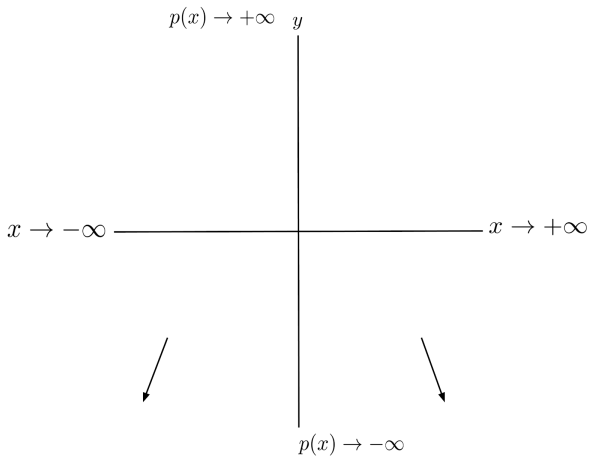
End Behavior

End behavior tells us whether the polynomial goes up or down as we move away from the origin.

For example, if the end behavior is $p(x) \to -\infty$ as $x \to +\infty$ and $p(x) \to +\infty$ as $x \to -\infty$, then we know that the polynomial goes down as we go right, and up as we go left.



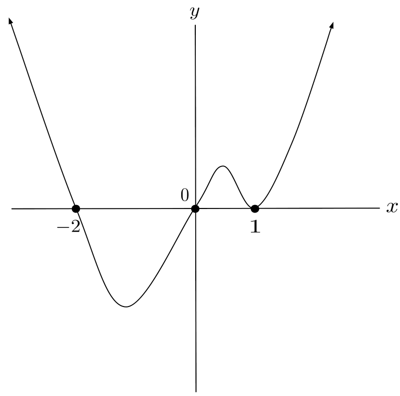
Similarly, if the end behavior is $p(x) \to -\infty$ as $x \to +\infty$ and $p(x) \to -\infty$ as $x \to -\infty$, then we know that the polynomial goes down as we go right, and down as we go left.



Zeros

The zeros tell us where the polynomial crosses the x-axis, and the factored form tells us whether the polynomial crosses or doubles back at each zero: if the exponent of the factor is odd, then the polynomial crosses; if the exponent of the factor is even, then the polynomial doubles back.

For example, if the factored form of polynomial is $p(x) = x(x+2)(x-1)^2$, then the polynomial crosses the x-axis at 0 and -2, and doubles back at 1. Combining this information with the end behavior, which is $p(x) \to +\infty$ as $x \to +\infty$ and $p(x) \to +\infty$ as $x \to -\infty$, we can draw a rough sketch of the polynomial.



Demonstration

Let's sketch a rough graph of the following polynomial:

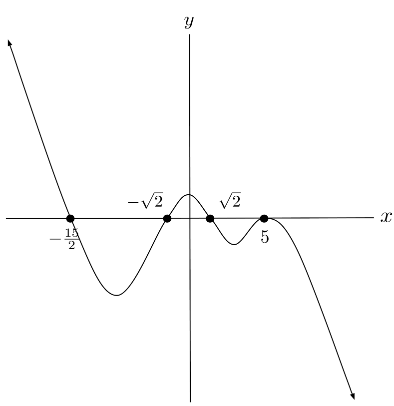
$$p(x) = -(2x + 15)(x + \sqrt{2})^3(x - \sqrt{2})^3(x - 5)^6$$

We first find the leading coefficient, $-(2x)(x)^3(x)^3(x)^6 = -2x^{13}$, which tells us the end behavior: $p(x) \to -\infty$ as $x \to +\infty$, and $p(x) \to +\infty$ as $x \to -\infty$.

Then, we can look at the factors and their exponents to find the zeros and tell whether the polynomial crosses the x-axis or doubles back at each zero.

Factor	**Zero**	**Cross or Double Back**
$(2x+15)$	$-\frac{15}{2}$	cross
$(x+\sqrt{2})^3$	$-\sqrt{2}$	cross
$(x-\sqrt{2})^3$	$\sqrt{2}$	cross
$(x-5)^6$	5	double back

To sketch the graph, we draw our end behavior, plot the zeros on the x-axis, and then connect them with the correct crossing or doubling back behavior.



Exercises

Sketch a rough graph of each polynomial.

1) $p(x) = (x-1)^3(x+2)^2(x-3)$

2) $p(x) = (x+5)^3(x-\sqrt{3})^2(x+\sqrt{3})^2$

3) $p(x) = -(2x+3)^3(x+5)(x-7)^8$

4) $p(x) = (x+2)(x-1)(3x-8)$

5) $p(x) = x^2(x+3)^2(x-3)^4$

6) $p(x) = -(4x+3)^{12}(3-x)^9(x-2)^{10}$

7) $p(x) = (3x-4)^{20}(8x+7)^{11}(x-1)^{31}(x-2)^{18}$

8) $p(x) = -x^{99}(2x-11)^{15}(x-3)(4x+7)^{32}(x-1)^2(x-7)^4$

Chapter 5
Rational Functions

5.1 Polynomial Long Division

A **rational function** is a fraction whose numerator and denominator are both polynomials. Rational functions are usually written in **proper form**, where the numerator is of a smaller **degree** than the denominator. (The degree of a polynomial is its highest exponent.)

Methods for Converting to Proper Form

Sometimes, we can convert to proper form simply by splitting up the fraction.

$$\frac{x+1}{x}$$
$$\frac{x}{x}+\frac{1}{x}$$
$$1+\frac{1}{x}$$

Other times, we can convert to proper form by factoring part of the numerator so that it cancels with the denominator.

$$\frac{x^2+3x+5}{x+1}$$
$$\frac{(x^2+3x+2)+3}{x+1}$$
$$\frac{(x+1)(x+2)+3}{x+1}$$
$$\frac{(x+1)(x+2)}{x+1}+\frac{3}{x+1}$$
$$x+2+\frac{3}{x+1}$$

We can also use synthetic division, a fast algorithm for division of a linear factor that was introduced in the previous part on polynomials.

To divide x^4+3x^3+2x-5 by $x+2$, we set up a template with -2 (the zero of $x+2$) on the far left, and the coefficients of $x^4+3x^3+0x^2+2x-5$ along the top row.

After filling in an initial 0, we repeatedly add down the columns, multiplying each result by -2 before placing it in the next column.

$$\begin{array}{r|rrrrr} -2 & 1 & 3 & 0 & 2 & -5 \\ & 0 & -2 & -2 & 4 & -12 \\ \hline & 1 & 1 & -2 & 6 & -17 \end{array}$$

The bottom row then tells us the coefficients and remainder in the proper form.

$$\frac{x^4+3x^3+2x-5}{x+2} = x^3+x^2-2x+6-\frac{17}{x+2}$$

However, synthetic division only works with linear factors, so what do we do when a factor isn't linear?

Polynomial Long Division

When faced with more complicated rational functions, we can turn to **polynomial long division**, which works the same way as the long division algorithm that's familiar from simple arithmetic.

To divide $x^7 - 3x^6 - 5x^3 + 1$ by $x^3 + 2$, we set up a template with $x^3 + 2$ on the outside and $x^7 - 3x^6 - 5x^3 + 1$ on the inside.

On the inside, we write out all coefficients, including those which are 0 (and thus aren't written in the condensed expression).

$$
\begin{array}{r|llllllll}
\cline{2-9}
x^3+2 & x^7 & -3x^6 & +0x^5 & +0x^4 & -5x^3 & +0x^2 & +0x & +1
\end{array}
$$

We begin by multiplying the divisor $x^3 + 2$ by x^4 to yield $x^7 + 2x^4$, which cancels the interior x^7 term when we subtract.

$$
\begin{array}{r|llllllll}
 & x^4 & & & & & & & \\
\cline{2-9}
x^3+2 & x^7 & -3x^6 & +0x^5 & +0x^4 & -5x^3 & +0x^2 & +0x & +1 \\
 & x^7 & & & 2x^4 & & & & \\
\cline{2-9}
 & & -3x^6 & +0x^5 & -2x^4 & -5x^3 & +0x^2 & +0x & +1
\end{array}
$$

Then, we multiply $x^3 + 2$ by $-3x^3$ to yield $-3x^6 - 6x^3$, which cancels the next interior term $-3x^6$ when we subtract.

We repeat this process until the degree of the leftover terms is less than the degree of x^3+2, in which case the leftover terms become the remainder and appear as the numerator in the remaining fraction.

$$
\begin{array}{r|cccccccc}
 & x^4 & -3x^3 & -2x & +1 & +\frac{4x-1}{x^3+2} & & & \\
\hline
x^3+2 & x^7 & -3x^6 & +0x^5 & +0x^4 & -5x^3 & +0x^2 & +0x & +1 \\
 & x^7 & & & 2x^4 & & & & \\
\cline{2-9}
 & & -3x^6 & +0x^5 & -2x^4 & -5x^3 & +0x^2 & +0x & +1 \\
 & & -3x^6 & & & -6x^3 & & & \\
\cline{3-9}
 & & & & -2x^4 & +x^3 & +0x^2 & +0x & +1 \\
 & & & & -2x^4 & & & -4x & \\
\cline{5-9}
 & & & & & +x^3 & +0x^2 & +4x & +1 \\
 & & & & & +x^3 & & & +2 \\
\cline{6-9}
 & & & & & & & +4x & -1
\end{array}
$$

The top row gives the result in proper form:

$$\frac{x^7-3x^6-5x^3+1}{x^3+2} = x^4 - 3x^3 - 2x + 1 + \frac{4x-1}{x^3+2}$$

Exercises

Find the proper form of each rational function.

1) $r(x) = \frac{8x^5+7x^2+6}{4x^2}$

2) $r(x) = \frac{-9x^9+3x^7-2x^5+x^3+1}{3x^4}$

3) $r(x) = \frac{x^2+2x+1}{x+3}$

4) $r(x) = \frac{6x^2-5x-4}{2x^2+3x+1}$

5) $r(x) = \frac{2x^5+3x^4+x^2-x+5}{x+2}$

6) $r(x) = \frac{-4x^4+x^2+4x-4}{2x-1}$

7) $r(x) = \frac{3x^4+20x^2+2x+22}{x^2+5}$

8) $r(x) = \frac{-3x^7+x^6-8x^5+5x^4-5x^3+3x^2+2}{x^3+2x-1}$

9) $r(x) = \frac{x^7-2x^6-x^5+2x^3-x^2-6x-3}{x^5+x+1}$

10) $r(x) = \frac{6x^8+4x^7+3x^5+2x^4-9x^3-6x^2+1}{3x^3+2x^2}$

11) $r(x) = \frac{x^{13}-4x^{10}+6x^9-17x^6+8x^4-15x^3-2x+14}{x^6-4x^3+2}$

12) $r(x) = \frac{-18x^{12}+2x^{11}+25x^9+4x^8-x^7+3x^6-23x^5+22x^4-x^3}{9x^4-x^3+x+1}$

5.2 Horizontal Asymptotes

Like polynomials, rational functions can have end behavior that goes to positive or negative infinity. However, rational functions can also have another form of end behavior in which they become flat, approaching (but never quite reaching) a horizontal line known as a **horizontal asymptote**.

Demonstration

For example, consider the rational function $r(x) = \frac{3x-2}{4x+1}$. As we input larger and larger numbers in the positive direction, the function output becomes closer and closer to 0.75.

$$r(10) = \frac{28}{41} \approx 0.683$$

$$r(100) = \frac{298}{401} \approx 0.743$$

$$r(1000) = \frac{2998}{4001} \approx 0.749$$

The same thing happens as we input larger and larger numbers in the negative direction: the function output becomes closer and closer to 0.75.

$$r(-10) = \frac{-32}{-39} \approx 0.821$$

$$r(-100) = \frac{-302}{-399} \approx 0.757$$

$$r(-1000) = \frac{-3002}{-3999} \approx 0.751$$

As a result, we say that the function r has a horizontal asymptote at $y = 0.75$.

Why Horizontal Asymptotes Occur

To understand why this happens, take a look at the function in proper form, $r(x) = \frac{3}{4} + \frac{11/4}{4x+1}$.

When we input a very large positive or negative number, remainder fraction's denominator becomes much larger than its numerator, causing the remainder fraction to shrink to 0.

On the other hand, the $\frac{3}{4}$ term persists, which causes the output to be close to $\frac{3}{4}$ or 0.75.

$$\begin{aligned} r(\text{large number}) &= \frac{3}{4} + \frac{11/4}{\text{large number}} \\ &\approx \frac{3}{4} + 0 \\ &= 0.75 \end{aligned}$$

Perhaps even more intuitively, notice that when we input very large values of x into $\frac{3x-2}{4x+1}$, the leading (highest degree) terms in the numerator and denominator become so much larger than the other terms, that the other terms cease to matter. The fraction then becomes approximately the ratio of the leading terms, $\frac{3x}{4x}$, which simplifies to $\frac{3}{4}$, or 0.75 in decimal form.

Case when the Denominator has Greater Degree

Now consider the case when the denominator is of a greater degree than the numerator -- say, when $r(x) = \frac{2x+1}{x^2-1}$.

Again, when we input very large values of x into the function, the leading terms in the numerator and denominator become the only terms that matter. The fraction then becomes approximately $\frac{2x}{x^2}$, which simplifies to $\frac{2}{x}$.

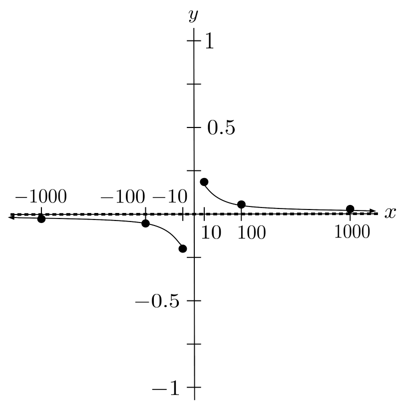
When we input very large values for x, the denominator becomes very large while the numerator stays the same, causing the fraction to shrink to 0.

As a result, the function has a horizontal asymptote at 0. We can confirm this by evaluating the function.

$$r(10) = \frac{21}{99} \approx 0.212 \qquad r(-10) = \frac{-19}{99} \approx -0.192$$

$$r(100) = \frac{201}{9999} \approx 0.020 \qquad r(-100) = \frac{-199}{9999} \approx -0.019$$

$$r(1000) = \frac{2001}{999999} \approx 0.002 \qquad r(-1000) = \frac{-1999}{999999} \approx -0.002$$



Case when the Numerator has Greater Degree

Lastly, consider a rational function whose numerator is of greater degree than its denominator -- say, $r(x) = \frac{3x^3+2x+1}{5x-2}$.

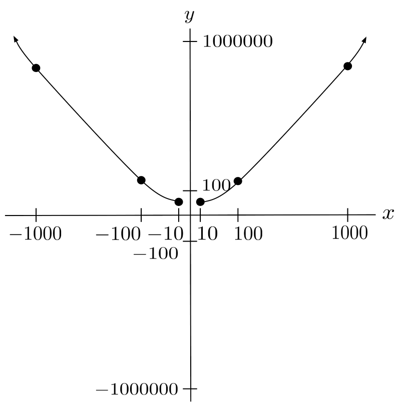
Taking the ratio of leading terms, we have $\frac{3x^3}{5x}$, which simplifies to $\frac{3}{5}x^2$. This expression grows without bound when we input large values of x, so the function has no horizontal asymptote.

We can confirm this by evaluating the function.

$$r(10) \approx 63 \qquad r(-10) \approx 58$$

$$r(100) \approx 6025 \qquad r(-100) \approx 5976$$

$$r(1000) \approx 600240 \qquad r(-1000) \approx 599760$$



Exercises

Find the horizontal asymptote, if any, of each rational function.

1) $r(x) = \frac{5x^3+3x-2}{x^3+2x^2-x}$ 2) $r(x) = \frac{6x^7+7x^6+x^2+1}{-3x^7+4x^5-x^3+2}$

3) $r(x) = \frac{x^5+5x-4}{2x^3+1}$ 4) $r(x) = \frac{-8x^7+4x^4-7x^2}{-7x^{10}+2x+1}$

5) $r(x) = \frac{(x^2+4)(3x^5+7x^2+2)}{(3x^4+5)(2x^3-x^2-3)}$

6) $r(x) = \frac{(2-3x^2)(x^3-2x^2+5)}{(2x+3)(x^2+1)(x^2-1)}$

7) $r(x) = \frac{(x+1)(x-2)(x+3)}{(x^2+7)(8+x^3)}$

8) $r(x) = \frac{(x+1)^3(2x^2+3x+4)}{(4x^2+7x-8)^2}$

9) $r(x) = \frac{(x+5)^2(2x+3)^3}{(3x^2-2)(3x^3+5)}$

10) $r(x) = \frac{x(1-2x)^3(x+2)}{(4x-1)(2-x)^4}$

5.3 Vertical Asymptotes

Unlike polynomials, rational functions can “blow up” to positive or negative infinity even for relatively small input values. Such input values are called **vertical asymptotes**, because they represent vertical lines that the function approaches but never quite reaches.

Demonstration

For example, consider the rational function $r(x) = \frac{3x+2}{x-5}$. As we input numbers closer and closer to 5 while staying greater than 5, the function output blows up to positive infinity.

$$r(5.1) = \frac{17.3}{0.1} = 173$$

$$r(5.01) = \frac{17.03}{0.01} = 1703$$

$$r(5.001) = \frac{17.003}{0.001} = 17003$$

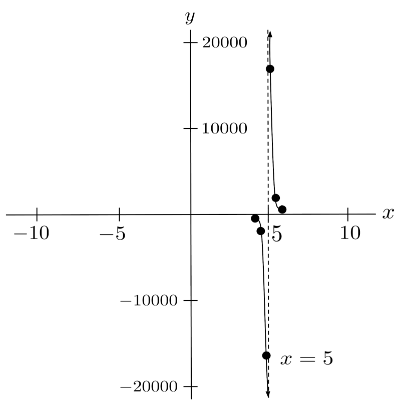
On the other hand, as we input numbers closer and closer to 5 while staying less than 5, the function output blows up to negative infinity.

$$r(4.9) = \frac{16.7}{-0.1} = -167$$

$$r(4.99) = \frac{16.97}{-0.01} = -1697$$

$$r(4.999) = \frac{16.997}{-0.001} = -16997$$

As a result, we say the function r has a vertical asymptote at $x = 5$.



To understand why this happens, notice that as our inputs become closer and closer to 5, the denominator becomes closer and closer to 0, while the numerator becomes closer and closer to 17.

As a result, we end up dividing a fairly constant numerator by a smaller and smaller denominator, which yields a bigger and bigger result.

When the input is greater than 5, the denominator is positive, which makes the result positive. When the input is less than 5, the denominator is negative, which makes the result negative.

Case of Multiple Vertical Asymptotes

In general, vertical asymptotes occur when the denominator is zero and the numerator is nonzero. In the above example, when we input 5, the denominator is 0, but the numerator is 17.

There can also be multiple vertical asymptotes -- for example, in the rational function $r(x) = \frac{2x+1}{x^2-4}$, inputting 2 makes the denominator 0 while the numerator is 5, and inputting -2 makes the denominator 0 while the numerator is -3.

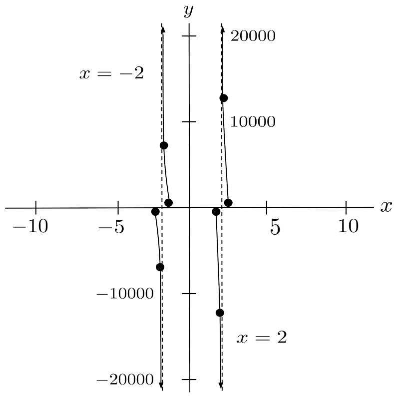
We confirm that $x = 2$ and $x = -2$ are indeed asymptotes by evaluating the function.

$$r(1.99) \approx -125 \qquad r(-1.99) \approx 75$$

$$r(1.9999) \approx -12500 \qquad r(-1.9999) \approx 7500$$

$$r(2.01) \approx 125 \qquad r(-2.01) \approx -75$$

$$r(2.0001) \approx 12500 \qquad r(-2.0001) \approx -7500$$



Case of No Vertical Asymptote

On the other hand, if the denominator is zero and the numerator is also zero, then the input is not necessarily a vertical asymptote of the function.

For example, inputting -1 to $r(x) = \frac{x^2-x-2}{x+1}$ makes the denominator 0, but it also makes the numerator 0, and the result is that the fraction does not blow up to infinity.

$$r(-1.01) = -3.01 \qquad r(-0.99) = -2.99$$

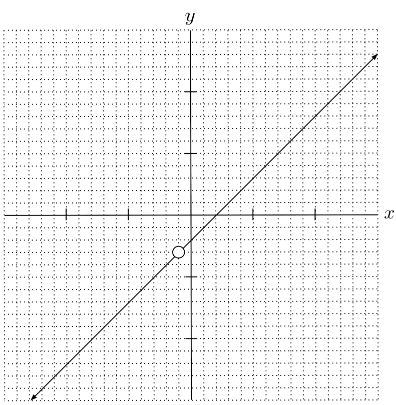
$$r(-1.0001) = -3.0001 \qquad r(-0.9999) = -2.9999$$

To understand this behavior, notice that provided x is not equal to -1, the function can simplify.

$$\begin{aligned} r(x) &= \frac{x^2 - x - 2}{x+1} \\ &= \frac{(x+1)(x-2)}{x+1} \\ &= x - 2 \end{aligned}$$

When we input an x that is not equal to 1, the $x+1$ factors in the numerator and denominator cancel each other out, and we are left with $x-2$.

As a result, the graph of r is just the graph of $y = x - 2$ with a hole at $x = -1$ (where it is undefined).



Exercises

Find the vertical asymptote(s), if any, of each rational function.

1) $r(x) = \frac{2x}{x-4}$

2) $r(x) = \frac{x^2-1}{3x+5}$

3) $r(x) = \frac{2x^2-8}{x-2}$

4) $r(x) = \frac{x+3}{x^2+3x+2}$

5) $r(x) = \frac{x^2+x-2}{x^2-x-6}$

6) $r(x) = \frac{x^2+2x-15}{(x^2-9)(x^2+x-12)}$

7) $r(x) = \frac{x^4-5x^2+1}{x^2-3x+2}$

8) $r(x) = \frac{x^2+7x+12}{x^3+x^2-16x-16}$

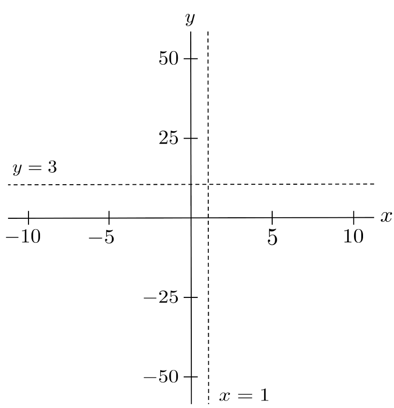
9) $r(x) = \frac{x^2-9}{x^3-3x^2+9x-27}$

10) $r(x) = \frac{2x-3}{(3x-2)(8x^2-2x-3)}$

5.4 Graphing with Horizontal and Vertical Asymptotes

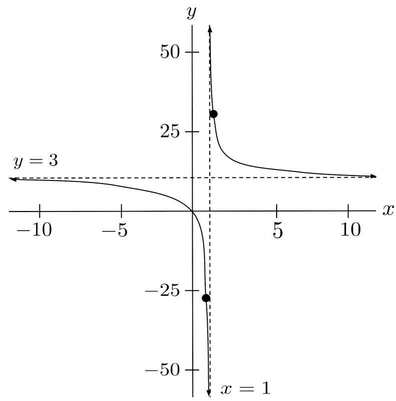
The horizontal and vertical asymptotes of a rational function can give us insight into the shape of its graph.

For example, consider the function $r(x)=\frac{3x}{x-1}$, which has a horizontal asymptote $y=3$ and a vertical asymptote $x=1$.



If we choose one input on each side of the vertical asymptote, we can tell which section of the plane the function will occupy.

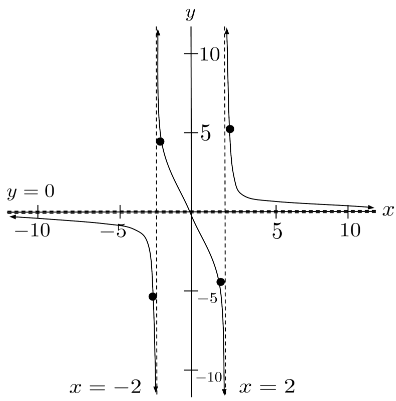
On the left side, we evaluate $r(0.9)=-27$, which indicates the section below the $y=3$ asymptote. On the right side, we evaluate $r(1.1)=33$, which indicates the section above the $y=3$ asymptote.



Case of Multiple Vertical Asymptotes

When there are multiple vertical asymptotes, we just have to choose test points on the sides of each asymptote.

For example, to graph the function $r(x)=\frac{x}{x^2-4}$ which has vertical asymptotes $x=-2$ and $x=2$, we can evaluate $r(-2.1)\approx -5.12$, $r(-1.9)\approx 4.87$, $r(1.9)\approx -4.87$, and $r(2.1)\approx 5.12$.



Exercises

Use horizontal and vertical asymptotes to graph the following rational functions.

1) $r(x) = \frac{3x-10}{x-4}$

2) $r(x) = \frac{x-4}{2x^2-9x+4}$

3) $r(x) = \frac{-3x^4+3x^2+x+32}{x^4-9}$

4) $r(x) = \frac{9x^5+1}{6x^5-54x^3}$

5) $r(x) = -\frac{x^3+x^2-x-1}{x^3-x^2-x+1}$

6) $r(x) = \frac{x+1}{x^4-20x^2+64}$

5.5 Graphing with Slant and Polynomial Asymptotes

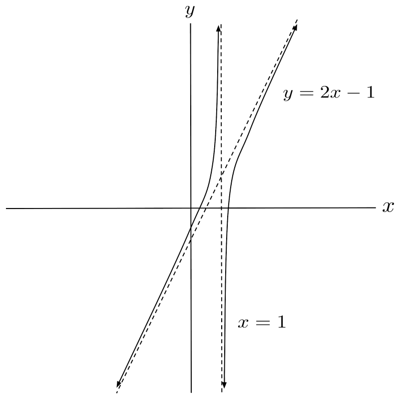
A horizontal asymptote is a horizontal line that arises from a constant whole number term in the proper form of a rational function.

Likewise, a **slant asymptote** is a slanted line that arises from a linear term in the proper form of a rational function.

Demonstration

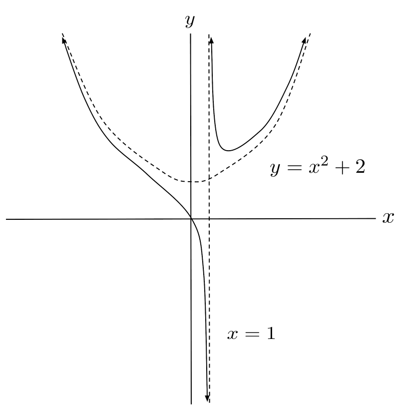
For example, the proper form of $r(x) = \frac{2x^2-3x}{x-1}$ is given by $r(x) = 2x - 1 - \frac{1}{x-1}$, which has $2x - 1$ as its whole number term.

As a result, r has a slant asymptote at $y = 2x - 1$, which appears in the graph of r below.



In general, the whole number part of the proper form is an asymptote. If the whole number part is of a higher degree, say $r(x) = \frac{x^3 - x^2 + 2x}{x - 1}$ with proper form $r(x) = x^2 + 2 + \frac{2}{x-1}$, then r has a **polynomial asymptote** at $y = x^2 + 2$.

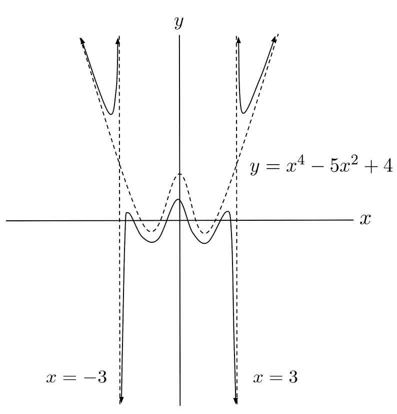
The graph of r approaches this asymptote just like it would approach any other horizontal or slant asymptote.



Existence and Degree

In general, a rational function has a horizontal, slant, or polynomial asymptote if the degree of the denominator is less than the degree of the numerator. The degree of the asymptote is given by the difference in degrees of the numerator and denominator.

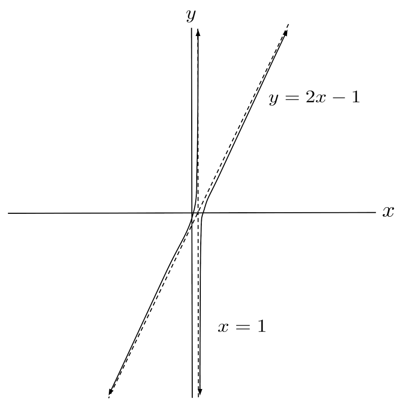
For example, $r(x) = \frac{x^6 - 14x^4 + 49x^2 - 30}{x^2 - 9}$ has a difference in degrees of $6 - 2 = 4$, so we should expect an asymptote of degree 4. Indeed, the proper form of the function is $r(x) = x^4 - 5x^2 + 4 + \frac{6}{x^2 - 9}$ which indicates a polynomial asymptote of $y = x^4 - 5x^2 + 4$.



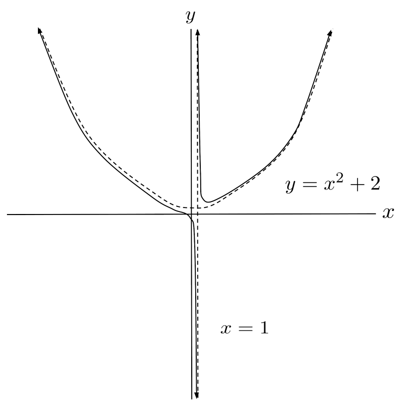
Big Picture

Zooming out of the previous graphs, we can see the big picture of rational functions: they look like their whole number part (i.e. their polynomial asymptotes), except at the **singularities** (vertical asymptotes), when the denominator of the fractional part becomes extremely small and the fraction blows up to positive or negative infinity.

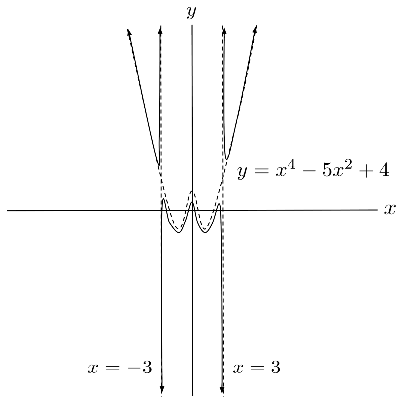
$$r(x) = 2x - 1 - \frac{1}{x-1}$$



$$r(x) = x^2 + 2 + \frac{2}{x-1}$$



$r(x) = x^4 - 5x^2 + 4 + \frac{6}{x^2-9}$



Exercises

Use vertical and horizontal/slant/polynomial asymptotes to graph the following rational functions.

1) $r(x) = \frac{4x^3-28x^2+56x-35}{2x^2-11x+12}$

2) $r(x) = \frac{4x^4+4x^3-26x^2-2x+13}{x^2+x-6}$

3) $r(x) = \frac{-x^4+10x^3+100}{x^3-5x^2-4x+20}$

4) $r(x) = \frac{-2x^4+5x^3+14x^2-30}{2x^2-3x-5}$

5) $r(x) = \frac{-x^6+7x^4+11x^3+5}{x^3-7x-6}$

6) $r(x) = \frac{-6x^6-7x^5+80x^4+70x^3-254x^2}{6x^2+7x-20}$

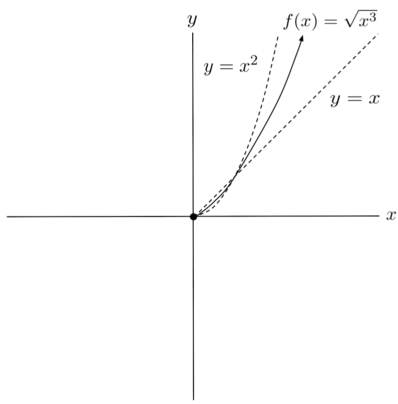
Part 6

Non-Polynomial Functions

6.1 Radical Functions

A **radical function** is a function that involves roots: square roots, cube roots, or any kind of fractional exponent in general. We can often infer what their graphs look like by sandwiching them between polynomial functions.

For example, the radical function $f(x) = \sqrt{x^3}$ can be written as $f(x) = x^{3/2}$, and its exponent $\frac{3}{2}$ is between 1 and 2, so the graph of f lies between the graphs of $y = x$ and $y = x^2$.



Negative Inputs

However, there is one caveat: $f(x) = \sqrt{x^3}$ is not defined for negative values of x. If we try to input a negative number, we end up taking the root of a negative number, which is undefined in the real numbers.

$$f(-4) = \sqrt{(-4)^3} = \sqrt{-64}$$

As a consequence, the graph of f remains blank for negative values of x, left of the y-axis.

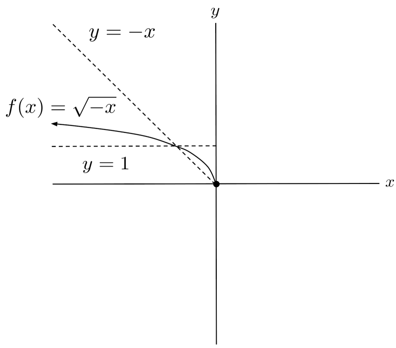
That being said, other radical functions can sometimes accept negative inputs, which are converted to positive numbers before the radical is applied.

For example, $x = -4$ is a valid input to $f(x) = \sqrt{-x}$ because the operation inside the root converts the negative input to a positive, and we can take the root of positive numbers.

$$f(-4) = \sqrt{-(-4)} = \sqrt{4} = 2$$

But the operation also converts positive inputs to negatives, so the positive section of the graph disappears.

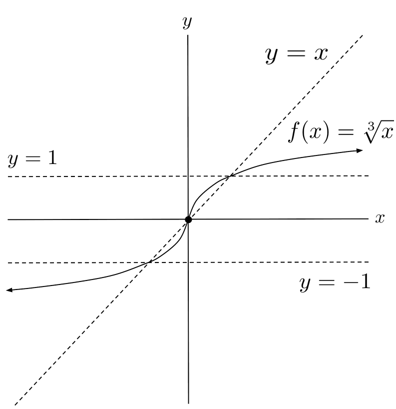
$$f(4) = \sqrt{-(4)} = \sqrt{-4}$$



Cube Root Functions

Unlike square root functions, cube root functions like $f(x) = \sqrt[3]{x}$ can accept both positive and negative inputs because cube roots are defined for both positive and negative numbers.

$$f(-8) = \sqrt[3]{-8} = -2$$
$$f(8) = \sqrt[3]{8} = 2$$



In general, whether a radical function covers the whole graph or just part of the graph depends on whether the root is an even root or an odd root.

- Even roots are NOT defined for negative numbers, so the graph is left blank for any input x that makes the inside of the root negative.
- Odd roots ARE defined for negative numbers, so the graph exists for any input x, even if it makes the inside of the root negative.

Just remember that whether an x-value is a valid input to a root function does not depend solely on the sign of the x-value, but rather on what the function does to the input x-value before applying the root.

Extraneous Solutions

When solving radical equations, valid algebraic steps can sometimes lead us to solutions that aren't actually correct.

For example, squaring both sides of the equation $\sqrt{x} = -2$ yields $x = 4$. However, when we input $x = 4$ into the equation to check the solution, we reach $\sqrt{4} = -2$, which simplifies to $2 = -2$, which is incorrect.

Therefore, we say that the solution $x = 4$ is **extraneous**, and the equation $\sqrt{x} = -2$ actually has no solutions in the real numbers.

Squaring both sides of an equation can introduce extraneous solutions because it introduces an additional solution that corresponds to the negative root.

It's easiest to see this if we forget about radicals for a moment -- for example, if we start with $x = 2$ and square both sides, we reach $x^2 = 4$, which is solved by $x = \pm\sqrt{4} = \pm 2$. Squaring both sides introduced a negative solution $x = -2$, and although $-2 = 2$ is not true, $(-2)^2 = 2^2$ is true. Likewise, although $x = 4$ is not a solution to $\sqrt{x} = -2$, it is a solution to $(\sqrt{x})^2 = (-2)^2$ because $(2)^2 = (-2)^2$.

A similar problem occurs when we raise both sides of an equation to the fourth, sixth, eighth, or any even power -- raising to an even power turns negative numbers to positives, so it introduces an additional solution that corresponds to the negative root.

On the other hand, raising both sides of an equation to the third, fifth, seventh, or any odd power does not change the sign of any numbers, so it won't lead to any extraneous solutions.

The main takeaway is that whenever we raise both sides of an equation to an even power, we need to double-check the solutions to make sure that they actually satisfy the equation.

Solving Radical Equations

In general, the best way to solve a complicated radical equation is to isolate the radical and exponentiate to cancel the radical.

Original equation	$2x - \sqrt[3]{2x^2 + x} = 0$
Isolate the radical	$\sqrt[3]{2x^2 + x} = 2x$
Cube both sides	$2x^2 + x = 8x^3$
Set polynomial equal to 0	$8x^3 - 2x^2 - x = 0$
Factor polynomial	$x(2x-1)(4x+1) = 0$
Solve	$x = 0, \frac{1}{2}, -\frac{1}{4}$
Remove extraneous solutions	$x = 0, \frac{1}{2}$

When there are multiple radicals in an equation, we first need to reduce the number of radicals in the equation until there is a single radical.

We can do this by repeatedly rearranging and exponentiating both sides of the equation.

Original equation	$\sqrt{x} + \sqrt{x-1} - \sqrt{x+1} = 0$
Rearrange	$\sqrt{x} + \sqrt{x-1} = \sqrt{x+1}$
Square	$x + 2\sqrt{x(x-1)} + x - 1 = x + 1$
Rearrange	$2\sqrt{x(x-1)} = 2 - x$
Square	$4x(x-1) = x^2 - 4x + 4$

Simplify	$3x^2 - 4 = 0$
Solve	$x = \pm\sqrt{\frac{4}{3}}$
Remove extraneous solutions	$x = \sqrt{\frac{4}{3}}$

Exercises

Graph the following radical functions.

1) $f(x) = \sqrt{x^5}$

2) $f(x) = \sqrt[6]{x^2}$

3) $f(x) = \sqrt[5]{x^3}$

4) $f(x) = \sqrt[5]{x^4}$

5) $f(x) = \sqrt[4]{-x^{15}}$

6) $f(x) = \sqrt[13]{x^{11}}$

Solve the following radical equations.

7) $\sqrt{x} + 1 = 2$

8) $2 - \sqrt[3]{x} = 5$

9) $\sqrt[4]{x^5} = -1$

10) $\sqrt{x} + x = 1$

11) $\sqrt{x^2} - 3 = x + 2$

12) $\sqrt[4]{3x - 2} = 2$

13) $\sqrt[7]{2x^2 + 3} = 3$

14) $\sqrt{x^2 - 4} - x = 2$

15) $\sqrt{x^2+2x}-x=x-1$

16) $\sqrt{2x+3}=\sqrt{3x+2}+1$

6.2 Exponential and Logarithmic Functions

Exponential functions have variables as exponents, e.g. $f(x) = 2^x$.

Their end behavior consists of growing without bound to infinity in one direction, and decaying to a horizontal asymptote of $y = 0$ in the other direction.

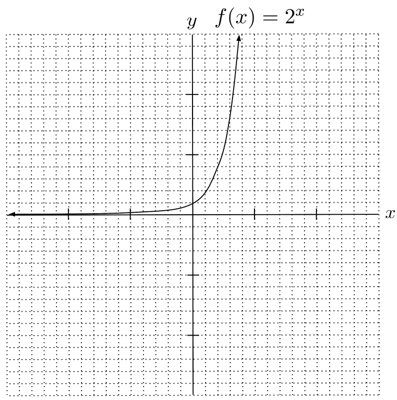
The size of the number that is exponentiated, called the **base**, governs which direction corresponds to which end behavior.

Exponential Growth

If the magnitude of the base is bigger than 1, then as x increases, the function is repeatedly multiplied by a number bigger than 1 and consequently grows without bound to infinity. For this reason, such functions are called **exponential growth** functions.

By the same token, as x decreases, the function is repeatedly divided by a number bigger than 1 and consequently decays to a horizontal asymptote of $y = 0$.

For example, for the exponential growth function $f(x) = 2^x$, each unit increase in x causes the output to be doubled, and each unit decrease in x causes the output to be halved.

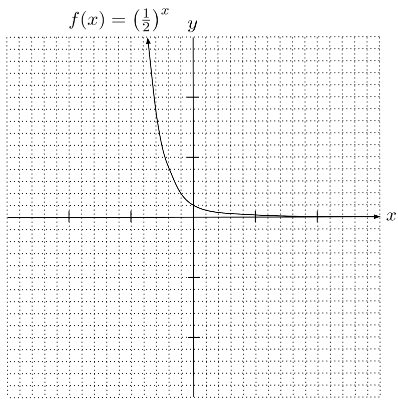


Exponential Decay

On the other hand, if the magnitude of the base is smaller than 1, then as x increases, the function is repeatedly multiplied by a number smaller than 1 and consequently decays to a horizontal asymptote of $y = 0$. For this reason, such functions are called **exponential decay** functions.

By the same token, as x decreases, the function is repeatedly divided by a number smaller than 1 and consequently grows without bound to infinity.

For example, for the exponential growth function $f(x) = \left(\frac{1}{2}\right)^x$, each unit increase in x causes the output to be halved, and each unit decrease in x causes the output to be doubled.



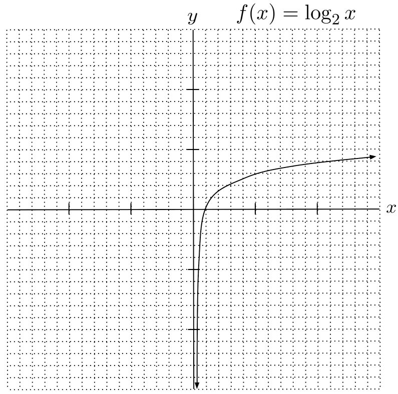
Logarithms

Equations involving exponential terms can be solved with the help of **logarithmic functions**, which cancel out exponentiation.

For example, the equation $2^x = 5$ is solved by $x = \log_2 5$, the logarithm base-2 of 5, which evaluates to roughly 2.32 via calculator.

If your calculator does not allow you to input a base for a logarithm, you can compute $\log_2 5$ as $\frac{\log 5}{\log 2}$. This is called the **change-of-base formula.**

Logarithmic graphs look similar to square-root graphs, except they cross the x-axis at 1 and extend downward towards an asymptote at $x = 0$.



Logarithmic graphs cross the x-axis at 1 because raising any number to the power of 0 results in 1. That is, any logarithm $x = \log_b 1$ solves the equation $b^x = 1$, which we already know is solved by $x = 0$.

Also, logarithmic graphs extend to negative infinity as x approaches 0, because a number (greater than one) gets smaller and smaller as its exponent gets more and more negative.

$$2^{-1} = \tfrac{1}{2} \qquad 2^{-2} = \tfrac{1}{4} \qquad 2^{-3} = \tfrac{1}{8}$$

$$\log_2 \tfrac{1}{2} = -1 \qquad \log_2 \tfrac{1}{4} = -2 \qquad \log_2 \tfrac{1}{8} = -3$$

Lastly, the base of the logarithm tells us where the y-value is 1 -- that is, the function $f(x) = \log_b x$ has $f(b) = 1$. This is because $\log_b b$ is the exponent we have to raise b to, to get b.

When the base of the logarithm is smaller than one, the graph flips over the x-axis.

In this case, the graph extends to positive infinity as x approaches 0, because a number smaller than 1 gets closer and closer to 0 as its exponent increases.

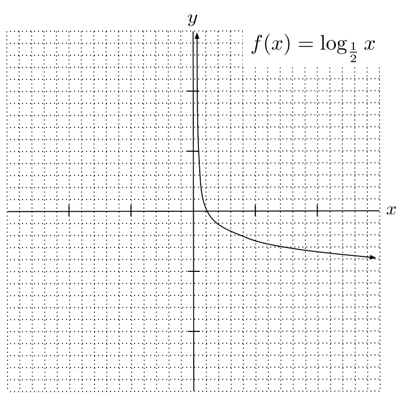
Likewise, as x increases, the graph becomes more and more negative because a negative exponent is needed to flip the fractional base.

$$\left(\tfrac{1}{2}\right)^1 = \tfrac{1}{2} \qquad \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4} \qquad \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8}$$

$$\log_{\frac{1}{2}} \tfrac{1}{2} = 1 \qquad \log_{\frac{1}{2}} \tfrac{1}{4} = 2 \qquad \log_{\frac{1}{8}} \tfrac{1}{2} = 3$$

$$\left(\tfrac{1}{2}\right)^{-1} = 2 \qquad \left(\tfrac{1}{2}\right)^{-2} = 4 \qquad \left(\tfrac{1}{2}\right)^{-3} = 8$$

$$\log_{\frac{1}{2}} 2 = -1 \qquad \log_{\frac{1}{2}} 4 = -2 \qquad \log_{\frac{1}{2}} 8 = -3$$



Properties of Logarithms

Expressions consisting of multiple logarithms of the same base can be simplified by using two properties of logarithms:

1. Addition outside two logarithms with the same base turns into multiplication inside a single logarithm. For example, $\log_2 4 + \log_2 8 = \log_2 32$, and in general, $\log_b x + \log_b y = \log_b xy$.
2. Multiplication outside two logarithms with the same base turns into exponentiation inside a single logarithm. For example, $3 \log_2 4 = \log_2 4^3 = \log_2 64$, and in general, $a \log_b x = \log_b x^a$.

A particularly noteworthy consequence of the second rule is that negative outside a log turns into reciprocal inside the log:

$$-\log_b x = (-1)\log_b x = \log_b x^{-1} = \log_b \frac{1}{x}$$

Additionally, logarithms of different bases can sometimes be converted to logarithms of the same base. For example, $\log_2 4$ is the same as $\log_4 16$. In general, $\log_{b^n} x^n = \log_b x$ provided both logarithms exist.

Below is an example of simplifying a logarithmic expression using all of the properties that we have discussed:

Original expression	$\log_2 x - \log_4 x$
Rewrite using addition	$\log_2 x + (-1)\log_4 x$
Convert multiplication to exponentiation	$\log_2 x + \log_4 x^{-1}$
Simplify	$\log_2 x + \log_4 \frac{1}{x}$
Square base and argument	$\log_{2^2} x^2 + \log_4 \frac{1}{x}$
Simplify	$\log_4 x^2 + \log_4 \frac{1}{x}$
Convert addition to multiplication	$\log_4 \left(x^2 \cdot \frac{1}{x}\right)$
Simplify	$\log_4 x$

Exercises

Graph the following exponential functions.

1) $f(x) = 3^x$

2) $f(x) = 5^x$

3) $f(x) = \left(\frac{1}{3}\right)^x$

4) $f(x) = \left(\frac{1}{5}\right)^x$

5) $f(x) = \left(\frac{3}{2}\right)^x$

6) $f(x) = \left(\frac{2}{3}\right)^x$

Use logarithms to solve the following exponential equations.

7) $3^x = 10$

8) $5^x = 7$

9) $\left(\frac{1}{3}\right)^x = \frac{1}{10}$

10) $\left(\frac{1}{5}\right)^x = \frac{1}{2}$

11) $\left(\frac{3}{2}\right)^x = 9$

12) $\left(\frac{2}{3}\right)^x = \frac{1}{5}$

Graph the following logarithmic functions. Use logarithm rules to simplify the expression, if needed.

13) $f(x) = \log_3 x$

14) $f(x) = \log_5 x$

15) $f(x) = \log_{\frac{1}{3}} x$

16) $f(x) = \log_{\frac{1}{5}} x$

17) $f(x) = \log_2 \sqrt{x}$

18) $f(x) = \log_{10} x^2 - \log_{10} x$

19) $f(x) = 3\log_3 1 + \frac{1}{2}\log_3 x$

20) $f(x) = \log_2 x - 2\log_4 \sqrt{x}$

6.3 Absolute Value

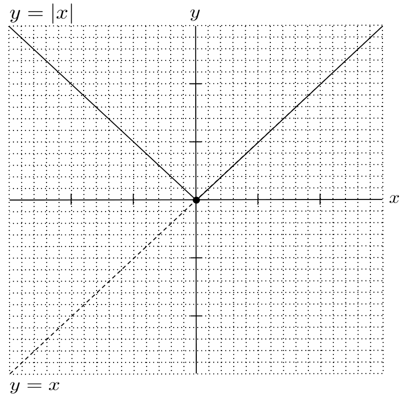
An **absolute value** function represents the **magnitude** of a number, i.e. its distance from 0.

For example, the absolute value of -3 is 3, and the absolute value of 4 is 4. We write this as $|-3| = 3$, and $|4| = 4$.

In effect, absolute value just removes the negative sign from a number, if there is a negative sign to begin with.

Graphs

Absolute value graphs are very straightforward -- they look similar to the graph of $y = x$, except the the outputs of negative x are turned positive.



Solving Equations by Splitting

Absolute value equations are similar to square root equations, in that we have to consider both positive and negative solutions. For example, the solutions to the equation $|x| = 2$ are $x = \pm 2$.

We can usually solve more complicated absolute value equations by isolating the absolute value and then breaking it up into positive and negative equations.

Original equation	$\lvert x^2 - 3\rvert - 1 = 0$
Isolate the absolute value	$\lvert x^2 - 3\rvert = 1$
Split into positive and negative equations	$x^2 - 3 = 1$ or $x^2 - 3 = -1$
Solve	$x = \pm 2, \pm\sqrt{2}$

Extraneous Solutions

One caveat to solving absolute value equations this way is that if the original equation tells us that the absolute value equals a negative number, we will get the same solutions as if it were a positive number, but none of them will be correct because absolute value can never have a negative output.

Original equation	$\lvert x^2 - 3 \rvert = -1$
Split into positive and negative equations	$x^2 - 3 = -1$ or $x^2 - 3 = 1$
Solve	$x = \pm\sqrt{2}, \pm 2$
Check solutions	$x = \pm\sqrt{2}$ $\lvert(\pm\sqrt{2})^2 - 3\rvert = -1$ $\lvert 2 - 3 \rvert = -1$ $\lvert -1 \rvert = -1$ $1 = -1 \quad \text{(invalid)}$ $x = \pm 2$ $\lvert(\pm 2)^2 - 3\rvert = -1$ $\lvert 4 - 3 \rvert = -1$ $\lvert 1 \rvert = -1$ $1 = -1 \quad \text{(invalid)}$
Remove extraneous solutions	no solution

Whenever an equation tells us that the output of some absolute value is a negative number, the equation will have no solution.

That being said, if an equation tells us that the output of some absolute value is a negative variable expression, the equation might have a solution, because the variable expression itself might be negative at times.

In these cases, it's usually best to solve the absolute value using the conventional method of splitting up into positive and negative equations, and then check the answers afterward to remove any extraneous solutions.

Original equation	$\lvert x^2 - 3 \rvert = -2x$
Split into positive and negative equations	$x^2 - 3 = -2x$ or $x^2 - 3 = 2x$
Solve	$x = \pm 1, \pm 3$
Check solutions	$x = 1$ $\lvert(1)^2 - 3\rvert = -2(1)$ $\lvert 1 - 3\rvert = -2$ $\lvert -2\rvert = -2$ $2 = -2 \quad \text{(invalid)}$ $x = -1$ $\lvert(-1)^2 - 3\rvert = -2(-1)$ $\lvert 1 - 3\rvert = 2$ $\lvert -2\rvert = 2$ $2 = 2 \quad \text{(valid)}$ $x = 3$ $\lvert(3)^2 - 3\rvert = -2(3)$ $\lvert 9 - 3\rvert = -6$ $\lvert 6\rvert = -6$ $6 = -6 \quad \text{(invalid)}$

	$x = -3$ $\lvert(-3)^2 - 3\rvert = -2(-3)$ $\lvert 9 - 3\rvert = 6$ $\lvert 6\rvert = 6$ $6 = 6 \quad \text{(valid)}$
Remove extraneous solutions	$x = -1, -3$

Case of Multiple Absolute Value Terms

When there are multiple absolute value terms, we need to split the equation into positive and negative equations for each absolute value term, one after the other.

Original equation	$\lvert x-1\rvert = \lvert x^2 - 1\rvert + 1$
Split into positive and negative equations	$\begin{cases} x - 1 = \lvert x^2 - 1\rvert + 1 \\ x - 1 = -\lvert x^2 - 1\rvert - 1 \end{cases}$
Isolate remaining absolute value	$\begin{cases} \lvert x^2 - 1\rvert = x - 2 \\ \lvert x^2 - 1\rvert = -x \end{cases}$

Split into positive and negative equations	$\begin{cases} x^2-1=x-2 \\ x^2-1=-x+2 \\ x^2-1=-x \\ x^2-1=x \end{cases}$
Simplify	$\begin{cases} x^2-x+1=0 \\ x^2+x-3=0 \\ x^2+x-1=0 \\ x^2-x-1=0 \end{cases}$
Solve	$\begin{cases} \text{no solution} \\ x=\frac{-1\pm\sqrt{13}}{2} \\ x=\frac{-1\pm\sqrt{5}}{2} \\ x=\frac{1\pm\sqrt{5}}{2} \end{cases}$
Combine solutions	$x=\frac{-1\pm\sqrt{13}}{2}, \frac{-1\pm\sqrt{5}}{2}, \frac{1\pm\sqrt{5}}{2}$
Remove extraneous solutions	$x=\frac{-1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

Exercises

Solve the following absolute value equations.

1) $|x-3|=4$

2) $|3+5x|-7=0$

3) $|x-4|+3=0$

4) $|x^2+1|+2=4$

5) $|x^2+2x|=8$

6) $|2x-2|=x$

7) $|2x^2+1|=x+1$

8) $|3-4x^2|+x=2$

9) $|x+1|=|x|$

10) $|2x-1|=1-|x|$

11) $|x^2-4|=|x|-2$

12) $|x^2-1|=|2x|-1$

13) $|x|+|x+1|=|x-1|$

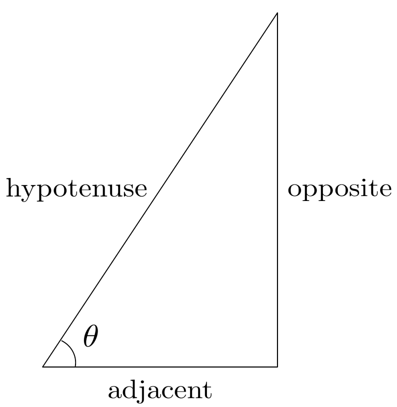
14) $|x|+|2x^2|=|x+1|$

6.4 Trigonometric Functions

Trigonometric functions represent the relationship between sides and angles in right triangles.

There are three main “trig” functions: sine, cosine, and tangent, and a mnemonic often used to remember what they represent is SohCahToa:

- The SINE of an angle is the ratio of the lengths of the OPPOSITE side and the HYPOTENUSE.
- The COSINE of an angle is the ratio of the lengths of the ADJACENT side and the HYPOTENUSE.
- The TANGENT of an angle is the ratio of the lengths of the OPPOSITE side and the ADJACENT side.

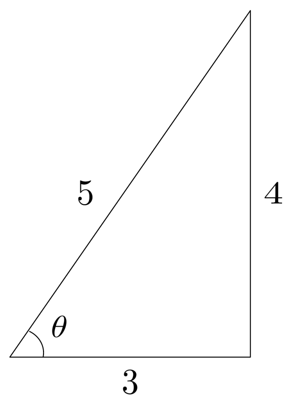


$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

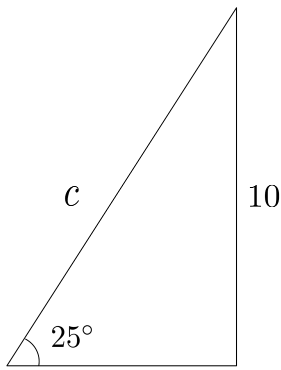
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

For example, in the triangle below, we have $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, and $\tan\theta = \frac{4}{3}$.



Solving for a Side

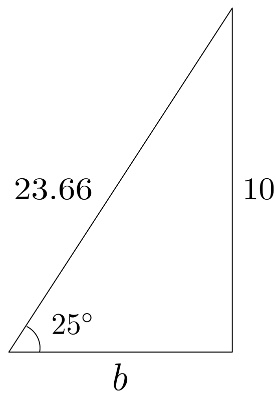
Trig functions can be used to solve for unknown side lengths in right triangles. For example, if we know that an angle is $25°$, the opposite side has a length of 10, and we want to find the hypotenuse, we can set up and solve an equation using sine.



Sine equation	$\sin 25° = \frac{10}{c}$
Solve	$c = \frac{10}{\sin 25°}$
Evaluate via calculator	$c \approx 23.66$

To find the remaining side, we can use any of three methods: Pythagorean theorem, cosine, or tangent.

No matter which technique we use, we will end up with the same result (though if we use our approximation of $c \approx 23.66$, we might be slightly off due to rounding error).

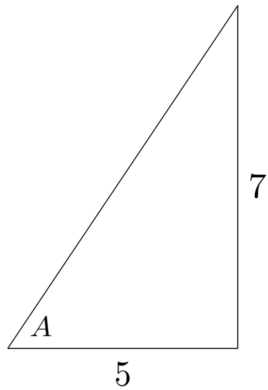


Pythagorean theorem	$10^2 + b^2 = 23.66^2$ $b = \sqrt{23.66^2 - 10^2}$ $b \approx 21.44$
Cosine	$\cos 25^\circ = \frac{b}{23.66}$ $b = 23.66 \cos 25^\circ$ $b \approx 21.44$
Tangent	$\tan 25^\circ = \frac{10}{b}$ $b = \frac{10}{\tan 25^\circ}$ $b \approx 21.45$

Solving for an Angle

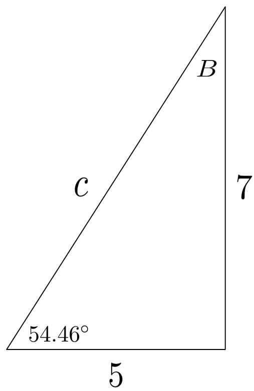
Similarly, using inverse trig functions, we can solve for unknown angles in right triangles.

For example, if we know that the adjacent side is 5 and the opposite side is 7, we can set up an equation with tangent and then use inverse tangent to find the angle.



Tangent equation	$\tan A = \frac{7}{5}$
Inverse tangent	$A = \tan^{-1}\left(\frac{7}{5}\right)$
Evaluate via calculator	$A \approx 54.46°$

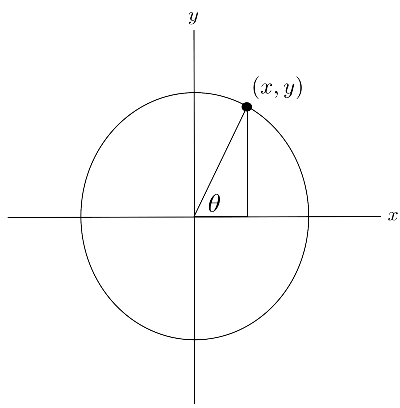
To find the remaining angle, we can use any of three methods: sum of degrees in a triangle, tangent, or Pythagorean theorem followed by sine or cosine. Regardless of which method we choose, we will end up with the same result.



Sum of degrees in triangle	$54.46^\circ + B + 90^\circ = 180^\circ$ $B \approx 35.54^\circ$
Inverse tangent	$\tan B = \frac{5}{7}$ $B = \tan^{-1}\left(\frac{5}{7}\right)$ $B \approx 35.54^\circ$
Pythagorean theorem	$7^2 + 5^2 = c^2$ $c = \sqrt{74}$
Inverse sine	$\sin B = \frac{5}{\sqrt{74}}$ $B = \sin^{-1}\left(\frac{5}{\sqrt{74}}\right)$ $B \approx 35.54^\circ$
Inverse cosine	$\cos B = \frac{7}{\sqrt{74}}$ $B = \cos^{-1}\left(\frac{7}{\sqrt{74}}\right)$ $B \approx 35.54^\circ$

The Unit Circle

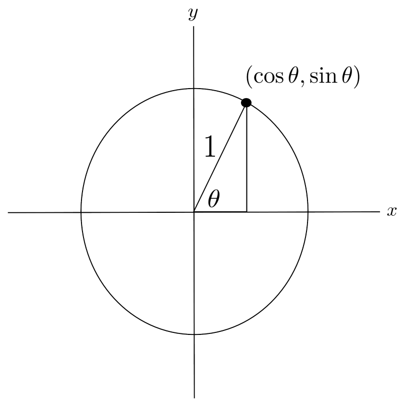
To gain a better understanding of trig functions, we can imagine putting a triangle inside of a circle on the coordinate plane.



The coordinates of the corner point (x, y) on the circle then tell us the other two sides of the triangle: the horizontal side has length x and the vertical side has length y. If we make the circle have radius 1, then the hypotenuse of the triangle is 1, and we have

$$\cos\theta = \tfrac{x}{1} = x$$
$$\sin\theta = \tfrac{y}{1} = y$$

and our point (x, y) can be written as $(\cos\theta, \sin\theta)$.



Immediately, we notice two important things. First, using the Pythagorean theorem on the triangle, we see that

$$\sin^2\theta + \cos^2\theta = 1.$$

This is a handy equation that can be useful in simplifying trigonometric expressions. For example, the expression $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$ is actually just equivalent to 2.

$$\begin{aligned}(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 &= (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta) \\ &\quad + (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta) \\ &= 2\sin^2\theta + 2\cos^2\theta \\ &= 2\end{aligned}$$

Second, angles repeat every 360°, since going 360° around the circle brings us back to the starting point of 0°.

That means, for example, that $\sin 390°$ and $\sin -330°$ are both equivalent to $\sin 30°$.

$$\begin{aligned} 390° &= 30° + 360° \\ -330° &= 30° - 360° \end{aligned}$$

Special Angles

For most angles, a calculator is needed to compute the corresponding trig function values. However, at particular angle measures, the trig functions have simple, exact values:

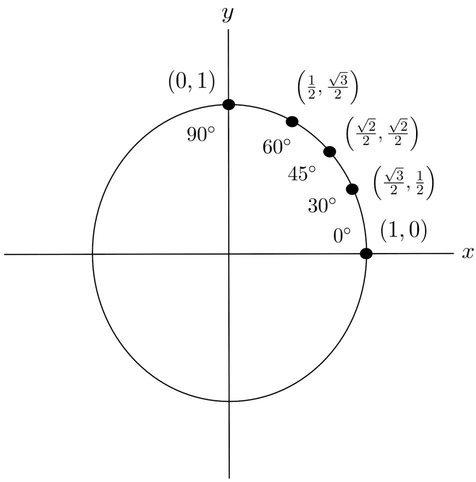
$$\begin{array}{lll} \sin 0° = 0 & \cos 0° = 1 & \tan 0° = 0 \\ \sin 30° = \frac{1}{2} & \cos 30° = \frac{\sqrt{3}}{2} & \tan 30° = \frac{\sqrt{3}}{3} \\ \sin 45° = \frac{\sqrt{2}}{2} & \cos 45° = \frac{\sqrt{2}}{2} & \tan 45° = 1 \\ \sin 60° = \frac{\sqrt{3}}{2} & \cos 60° = \frac{1}{2} & \tan 60° = \sqrt{3} \\ \sin 90° = 1 & \cos 90° = 0 & \tan 90° = \text{undefined} \end{array}$$

We can remember which values correspond to which angles and which trig functions by thinking about them visually in the unit circle and mentally pairing $\frac{\sqrt{3}}{2}$ with $\frac{1}{2}$.

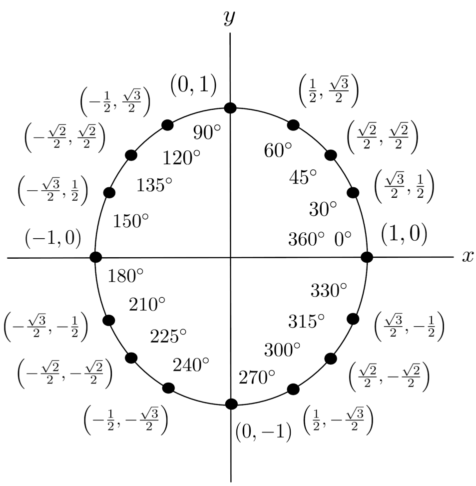
- At $30°$, the x-coordinate is bigger than the y-coordinate, so the x-coordinate must be $\frac{\sqrt{3}}{2}$ and the y-coordinate must be $\frac{1}{2}$.
- At $60°$, this is reversed.

- At $45°$, the x-coordinate and y-coordinate are the same, so they both are $\frac{\sqrt{2}}{2}$.
- At $0°$, we're on the x-axis, so the x-coordinate is 1 and the y-coordinate is 0.
- At $90°$, we're on the y-axis, so the y-coordinate is 1 and the x-coordinate is 0.

To get tangent, we can just take the ratio of the y-coordinate to the x-coordinate.



Using symmetry, we can label angles in the other three quadrants of the circle.

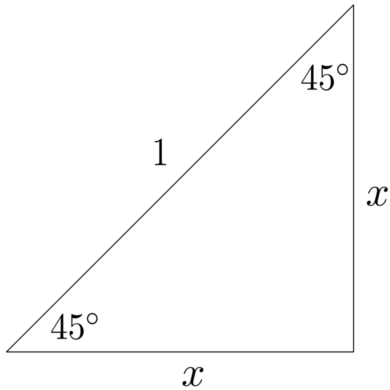


Derivation of Special Angles

You might be wondering where the values $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{3}}{2}$ come from in the first place.

To see where $\frac{\sqrt{2}}{2}$ comes from, we can construct a right triangle with a hypotenuse of 1 and an angle of 45°.

The other angle must also be 45°, so the triangle's two legs must be equal in length, and we can use the Pythagorean theorem to discover that the length of each leg is $\frac{\sqrt{2}}{2}$.

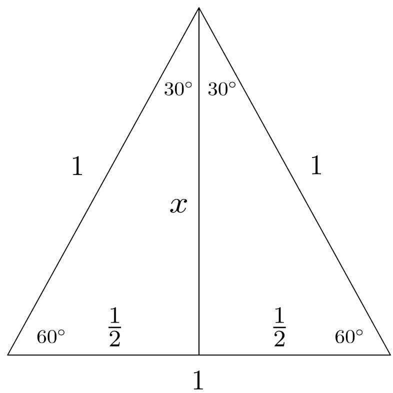


Pythagorean theorem	$x^2 + x^2 = 1^2$
Simplify	$2x^2 = 1$
Solve	$x^2 = \frac{1}{2}$ $x = \sqrt{\frac{1}{2}}$
Simplify	$x = \frac{1}{\sqrt{2}}$ $x = \frac{\sqrt{2}}{2}$

Likewise, to see where $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ come from, we can construct a right triangle with a hypotenuse of 1 and an angle of 60°.

The other angle must be 30°, which is exactly half -- consequently, we can combine two of these triangles to form an equilateral triangle whose side lengths are all equal to the hypotenuse of 1.

The shortest sides of the two triangles together make up a side of the equilateral triangle, which we know has length 1, so the shortest sides of the two triangles must each be $\frac{1}{2}$. Using the Pythagorean theorem, we find that the length of the other leg is $\frac{\sqrt{3}}{2}$.



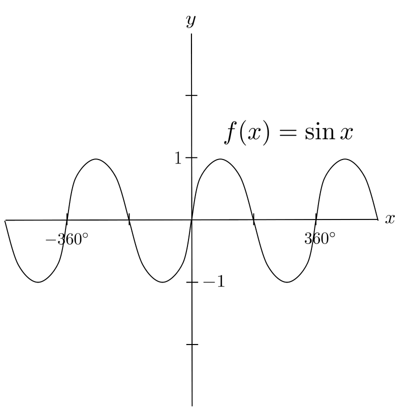
Pythagorean theorem	$\left(\frac{1}{2}\right)^2 + x^2 = 1^2$
Simplify	$\frac{1}{4} + x^2 = 1$
Solve	$x^2 = \frac{3}{4}$
	$x = \sqrt{\frac{3}{4}}$
Simplify	$x = \frac{\sqrt{3}}{2}$

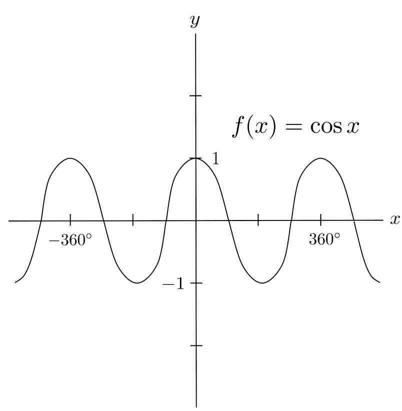
Graphs

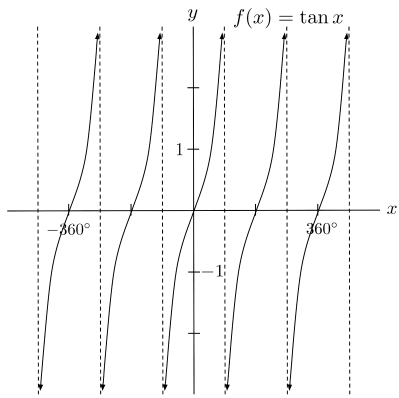
The graphs of sine, cosine, and tangent are drawn below. They repeat every $360°$, since $360°$ is one full revolution around the unit circle and thus brings us full-circle back to the starting point.

Tangent actually repeats twice every $360°$ (or once every $180°$) because it goes from positive to negative from the first to second quadrant, and again positive to negative from the third to fourth quadrant.

To make sense of the shapes of the graphs, try to trace out the trig function values while following around the unit circle.

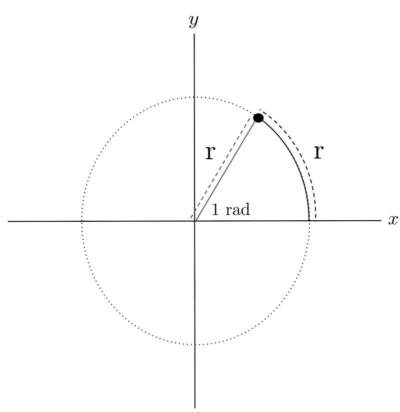






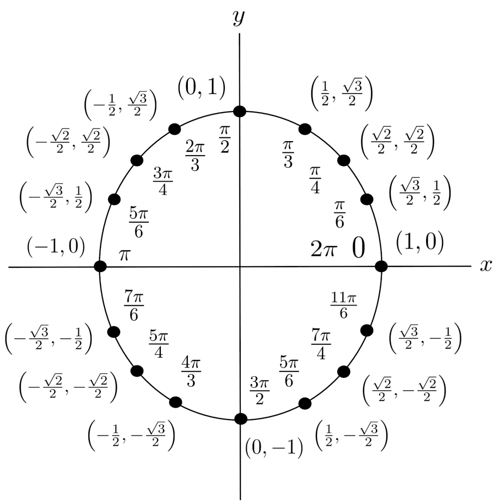
Radians

The standard way to measure angles is actually not in degrees -- rather, it is in **radians**. One radian is equivalent to the angle whose arc is equal to one radius of a circle.

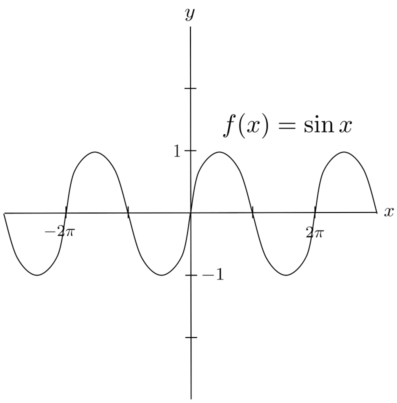


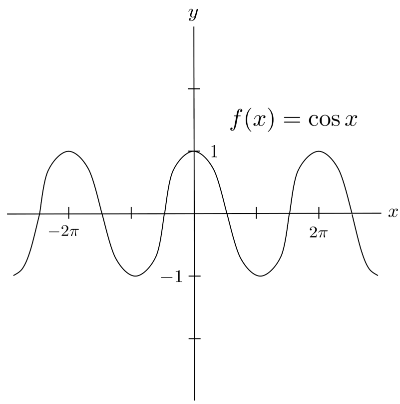
Since the full arc length (**circumference**) of the circle is 2π times the radius, a full 360° around the circle is equivalent to 2π radians.

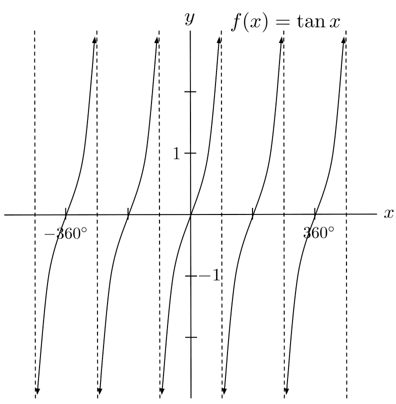
Below is a copy of the unit circle, using radians instead of degrees.



The trig functions are graphed in terms of radians below. Nothing changes, except for the units of the x-axis.



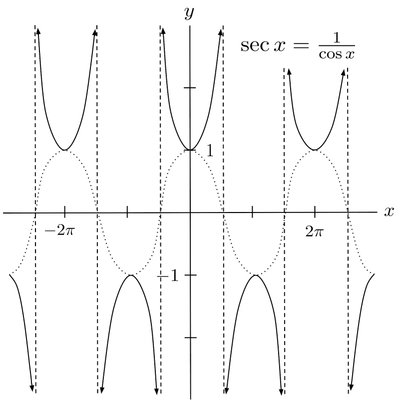


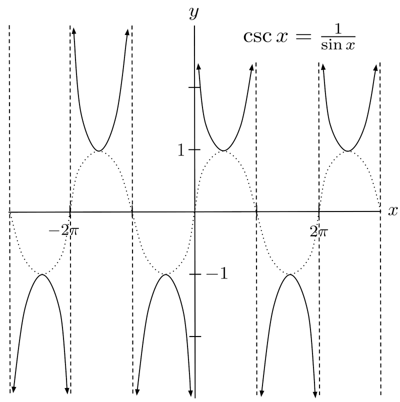


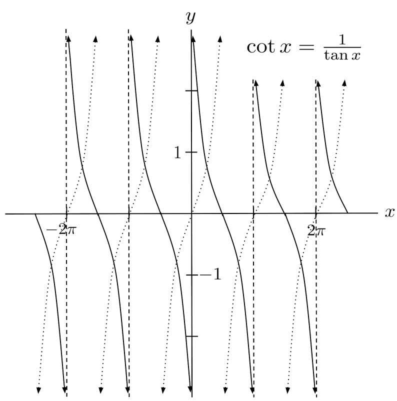
Reciprocal Trigonometric Functions

There are three other trig functions: secant, cosecant, and cotangent. They are just the reciprocals of cosine, sine, and tangent.

Consequently, they can be understood by thinking about the properties of cosine, sine, and tangent. We will not explore them further, but we include their graphs below.



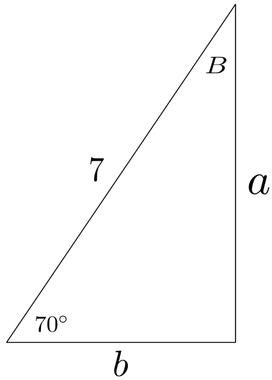




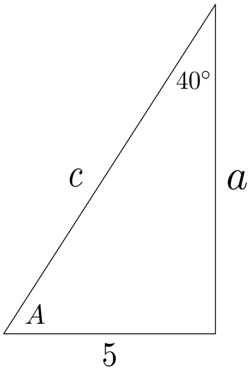
Exercises

Use trigonometry to find the missing sides and angles of the triangles.

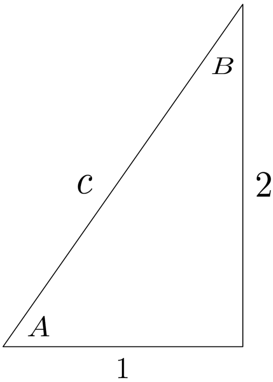
1)



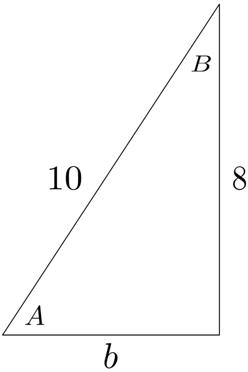
2)



3)



4)



Use the unit circle to find the exact values of the following trigonometric expressions.

5) $\cos 0°$

6) $\sin 270°$

7) $\sin 60°$

8) $\tan 30°$

9) $\cos 135°$

10) $\tan 270°$

11) $\sin 135°$

12) $\cos 240°$

13) $\sin 720°$

14) $\cos 480°$

15) $\tan 765°$

16) $\cos -45°$

17) $\sin -150°$

18) $\cos \frac{\pi}{3}$

19) $\sin \frac{5\pi}{6}$

20) $\tan \frac{2\pi}{3}$

21) $\cos \pi$

22) $\tan \frac{7\pi}{6}$

23) $\sin 2\pi$

24) $\cos 5\pi$

25) $\tan \frac{9\pi}{4}$

26) $\cos \frac{5\pi}{2}$

27) $\sin -\frac{3\pi}{2}$

28) $\cos -\frac{\pi}{6}$

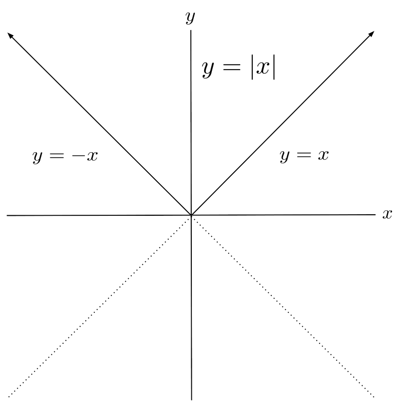
29) $\tan -\frac{\pi}{3}$

30) $\tan -\frac{5\pi}{4}$

6.5 Piecewise Functions

A **piecewise function** is pieced together from multiple different functions.

For example, the absolute value function is a piecewise function because it consists of the line $y = -x$ for negative x, and $y = x$ for positive x.



Case Notation

More generally, piecewise functions can be defined using case notation, which tells which functions to use as pieces and where to use them as pieces.

The absolute value function, for example, can be written in case notation as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

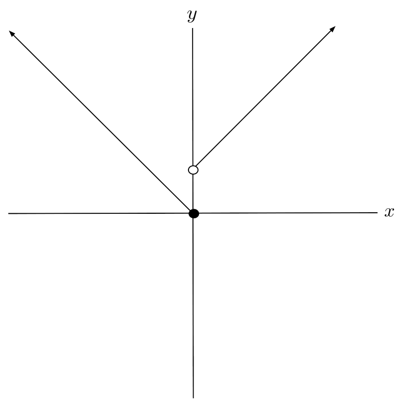
This case notation just tells us that for negative inputs ($x < 0$) we should use the function $y = -x$ to calculate the function output, and for nonnegative inputs ($x \geq 0$) we should use the function $y = x$ to calculate the function output.

Two more equivalent case notation forms for the absolute value function are shown below.

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases} \qquad |x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Sometimes, piecewise functions have breaks in them. For example, if we modify the case notation of the absolute value function so that the right piece is elevated, the graph has a break in it. This looks unusual, but it is a perfectly valid function.

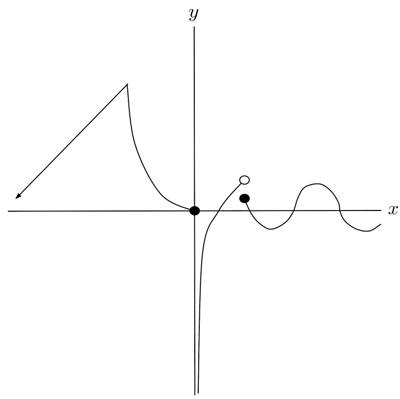
$$f(x) = \begin{cases} x+3 & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}$$



Many Function Types

There is no limit to what types of functions a piecewise function can consist of. For example, the equation and graph of a more complicated piecewise function are shown below.

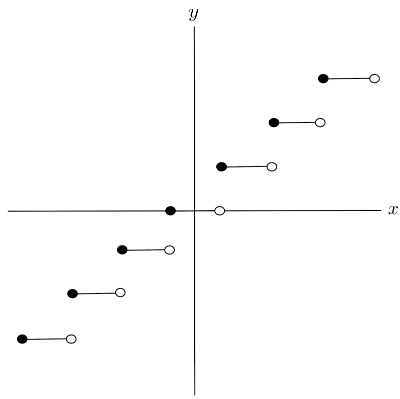
$$f(x) = \begin{cases} \sin x & \text{if } x \geq 2 \\ \log_2 x & \text{if } 0 < x < 2 \\ x^2 & \text{if } -2 < x \leq 0 \\ x+8 & \text{if } x \leq -2 \end{cases}$$



Many Cases

Likewise, there is no limit to the number of pieces a piecewise function can have. For example, rounding is an example of a piecewise function with infinitely many pieces.

$$f(x) = \begin{cases} \vdots & \vdots \\ 2 & \text{if } 1.5 \leq x < 2.5 \\ 1 & \text{if } 0.5 \leq x < 1.5 \\ 0 & \text{if } -0.5 \leq x < 0.5 \\ -1 & \text{if } -1.5 \leq x < -0.5 \\ -2 & \text{if } -2.5 \leq x < -1.5 \\ \vdots & \vdots \end{cases}$$



Exercises

Graph the following piecewise functions.

1) $$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ \frac{1}{2}x & \text{if } x < 0 \end{cases}$$

2) $$f(x) = \begin{cases} x - 5 & \text{if } x \geq 2 \\ x + 5 & \text{if } x < 2 \end{cases}$$

3) $$f(x) = \begin{cases} x^2 - 4 & \text{if } x \geq 2 \\ 2 - x & \text{if } 0 < x < 2 \\ x^2 - 4 & \text{if } x \geq 0 \end{cases}$$

4) $$f(x) = \begin{cases} \sin x & \text{if } x > 0 \\ -2x & \text{if } -2 < x \leq 0 \\ 8 - x^2 & \text{if } x \leq -2 \end{cases}$$

5) $$f(x) = \begin{cases} 2 - x^2 & \text{if } x \geq 1 \\ x^2 & \text{if } 0 < x < 1 \\ -1 & \text{if } x = 0 \\ x^2 & \text{if } -1 \leq x < 0 \\ 8 - x^2 & \text{if } x < -1 \end{cases}$$

6) $$f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ \log_2 x & \text{if } 0 < x \leq 1 \\ \cos x & \text{if } -\pi \leq x \leq 0 \\ 4 + x & \text{if } x < -\pi \end{cases}$$

Chapter 7

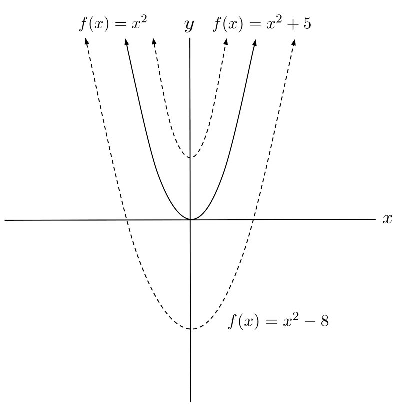
Transformations of Functions

7.1 Shifts

When a function is **shifted**, all of its points move vertically and/or horizontally by the same amount. The function's size and shape are preserved -- it is just slid in some direction, like sliding a book across a table.

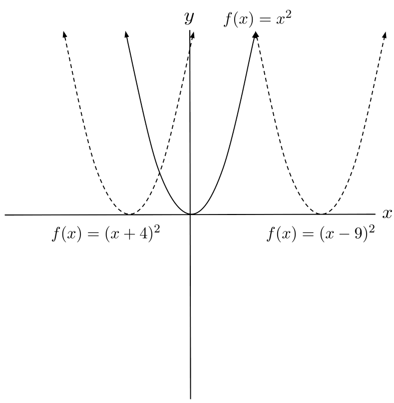
Shifts Outside the Function

Shifts occur when a constant term is added in a function. When the constant term is added on the outside of a function, e.g. when $f(x) = x^2$ is transformed into $f(x) = x^2 + 5$, the function shifts up by that many units. (If a negative term is added, the function moves down.)



Shifts Inside the Function

On the other hand, when the constant term is added on the inside of a function, e.g. when $f(x) = x^2$ is transformed into $f(x) = (x+4)^2$, the function shifts left by that many units. (If a negative term is added, the function moves right.)



Intuition

Vertical shifts are very intuitive: if we add a number to a function, that number is added to every output of the function. If the number is positive, every output y-value is increased by that amount. If the number is negative, every output y-value is decreased by that amount.

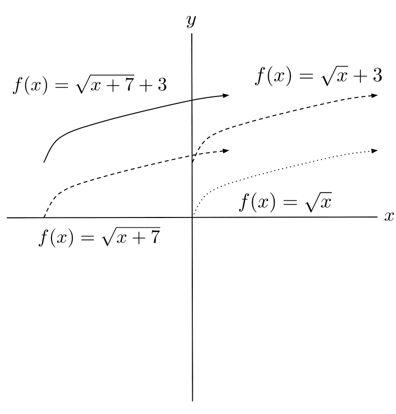
The intuition behind horizontal shifts is a little less straightforward, because ADDING a number inside a function moves it left in the NEGATIVE direction along the x-axis.

But think about it this way: when we transform $f(x) = x^2$ into $f(x) = (x+4)^2$, the output originally at $x = 4$ is now at $x = 0$, because 4^2 is the same as $(0+4)^2$. Similarly, the output originally at $x = 0$ is now at $x = -4$, because 0^2 is the same as $(-4+4)^2$. Every input needs to move 4 units left, to keep its output the same.

Combining Shifts

When we have both vertical and horizontal shifts, it doesn't matter which we perform first.

For example, to transform $f(x) = \sqrt{x}$ into $f(x) = \sqrt{x+7} + 3$, we can either shift it left 7 units and then up 3 units, or up 3 units and then left 7 units. Either way, we get the same result.



Exercises

Use shifts to graph the following functions.

1) $f(x) = x^2 + 3$

2) $f(x) = (x+8)^2$

3) $f(x) = x^3 - 5$

4) $f(x) = (x-4)^2$

5) $f(x) = \sqrt{x-3} + 7$

6) $f(x) = \sqrt[3]{x+4} + 6$

7) $f(x) = 2^{x+2} - 4$

8) $f(x) = \log_2(x+5) - 3$

9) $f(x) = \sin(x-2) - 4$

10) $f(x) = \tan(x-7) + 3$

11) $f(x) = |x+4| - 7$

12) $f(x) = |x-8| - 2$

7.2 Rescalings

When a function is **rescaled**, it is stretched or compressed along one of the axes, like a slinky. The function's general shape is preserved, but it might look a bit thinner or fatter afterwards.

Rescalings Outside the Function

Rescalings occur when a constant term is multiplied in a function. When the constant term is multiplied on the outside of a function, the function stretches or compresses along the y-axis.

For example, multiplying outside by 3 with the transformation

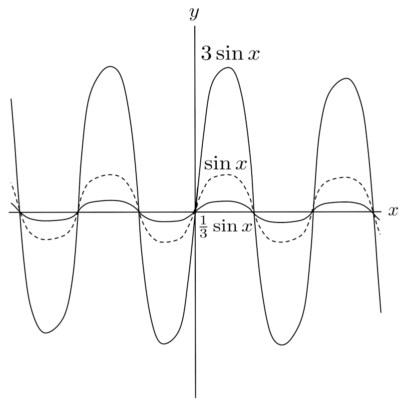
$$f(x) = \sin x \to f(x) = 3\sin x$$

stretches the function outward vertically, away from the x-axis.

On the contrary, multiplying outside by $\frac{1}{3}$ with the transformation

$$f(x) = \sin x \to f(x) = \frac{1}{3}\sin x$$

compresses the function inward vertically, towards the x-axis.



Rescalings Inside the Function

On the other hand, when the constant term is multiplied on the inside of a function, the function stretches or compresses horizontally along the x-axis.

For example, multiplying inside by 3 with the transformation

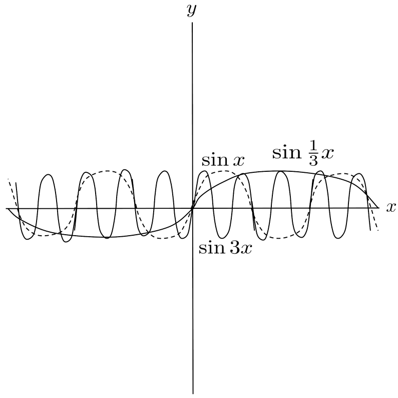
$$f(x) = \sin x \to f(x) = \sin 3x$$

compresses the function inward horizontally, towards the y-axis.

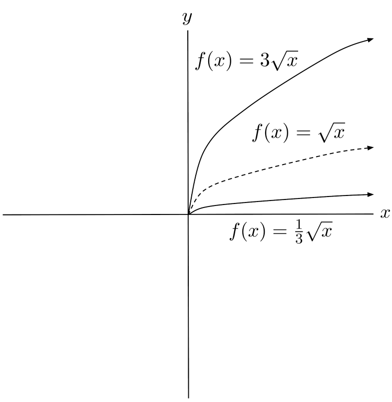
On the contrary, multiplying inside by $\frac{1}{3}$ with the transformation

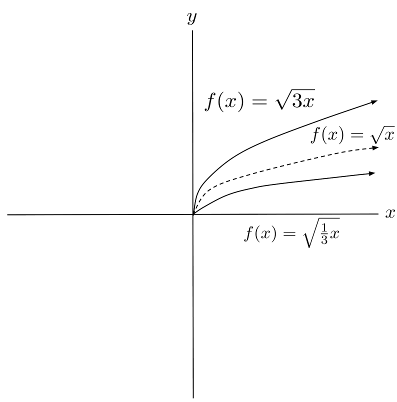
$$f(x) = \sin x \to f(x) = \sin \tfrac{1}{3}x$$

stretches the function outward horizontally, away from the y-axis.



For functions that are more linear than curvy, such as $f(x) = \sqrt{x}$, vertical and horizontal rescalings can have similar effects on the graph.





Intuition

Similar to vertical shifts, vertical rescalings are very intuitive: if we multiply a function by a number, every output of the function is multiplied by that number.

If the number is greater than 1, every output y-value is increased by the multiplier. If the number is less than 1, every output y-value is decreased by the multiplier.

Similar to horizontal shifts, the intuition behind horizontal rescalings is not as straightforward. Multiplying a BIG number inside a function COMPRESSES it, rather than stretching it.

Think about it this way: when we transform $f(x)=\sqrt{x}$ into $f(x)=\sqrt{3x}$, the output originally at $x=3$ is now at $x=1$, because $\sqrt{3}$ is the same as $\sqrt{3(1)}$. Similarly, the output originally at $x=1$ is now at $x=\frac{1}{3}$, because $\sqrt{1}$ is the same thing as $\sqrt{3\left(\frac{1}{3}\right)}$. Every input needs to be divided by 3, to keep its output the same.

Combining Rescalings and Shifts

When we have both vertical and horizontal rescalings, it doesn't matter which we perform first.

However, when dealing with rescalings and shifts simultaneously, it's important to perform horizontal shifts first, then rescalings, and lastly vertical shifts. This way, horizontal shifts are themselves rescaled, and vertical shifts are not.

To see why horizontal shifts themselves need to be rescaled, consider the function transformation of $f(x) = x$ into $f(x) = \sqrt{3x - 1}$.

In the original function, we have $\sqrt{1} = 1$. If we rescale first and then shift 1 right, then the input $x = 1$ is rescaled to $x = \frac{1}{3}$ and shifted to $x = \frac{4}{3}$.

When we input the transformed input into the transformed function, it should produce the same result as the original input in the original function -- but this is not the case for $x = \frac{4}{3}$.

$$\sqrt{3\left(\frac{4}{3}\right) - 1} = \sqrt{3} \neq 1$$

On the other hand, if we first shift 1 right and then rescale, then the input $x = 1$ is shifted to $x = 2$ and rescaled to $x = \frac{2}{3}$.

Indeed, $x = \frac{2}{3}$ produces the same result as the original input in the original function.

$$\sqrt{3\left(\frac{2}{3}\right) - 1} = \sqrt{1} = 1$$

Exercises

Use rescalings (followed by shifts) to graph the following functions.

1) $f(x) = 3x^2$

2) $f(x) = \left(\frac{1}{2}x\right)^2$

3) $f(x) = \frac{1}{4}x^3$

4) $f(x) = (2x)^3$

5) $f(x) = 4\sqrt{x}$

6) $f(x) = \sqrt[3]{5x}$

7) $f(x) = 4^{\frac{1}{2}x}$

8) $f(x) = 3\left(\frac{1}{2}\right)^x$

9) $f(x) = 4\sin x$

10) $f(x) = \tan\left(\frac{1}{3}x\right)$

11) $f(x) = \left|\frac{1}{2}x + 3\right|$

12) $f(x) = 4|x-2| - 7$

13) $f(x) = 2(4)^{\frac{1}{2}x-2}$

14) $f(x) = 8\cos\left(\frac{1}{4}x + 2\right) + 3$

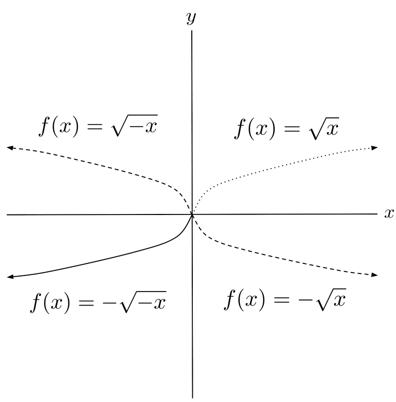
15) $f(x) = 5\sqrt[3]{7x-3} - 5$

16) $f(x) = \frac{1}{8}(4x-4)^2 - 5$

7.3 Reflections

When a function is **reflected**, it flips across one of the axes to become its mirror image.

Reflections occur when a function is made negative -- when the negative is outside the function, the reflection is over the y-axis; and when the negative is inside the function, the reflection is over the x-axis.



The intuition behind reflections is that, depending where it is placed, the negative sign switches positive and negative values of the x or y variable.

If the negative is outside the function, then the output y-value switches sign, essentially reflecting every point over the x-axis.

On the other hand, if the negative is inside the function, then the input x-value switches sign, essentially reflecting every point over the y-axis.

Order of Function Transformations

When we have both vertical and horizontal reflections, it doesn't matter which we perform first. Likewise, when dealing with reflections and rescalings simultaneously, it doesn't matter which we perform first.

However, when dealing with reflections and shifts simultaneously, it's important to perform horizontal shifts first, then reflections, and lastly vertical shifts.

We are left with an **order of function transformations**, similar to the concept of order of operations in arithmetic, but different in actual order:

1. Horizontal shifts
2. Rescalings and reflections (interchangeable)
3. Vertical shifts

Exercises

Use reflections and rescalings (followed by shifts) to graph the following functions.

1) $f(x) = \sqrt[3]{-x}$

2) $f(x) = -\cos(-x)$

3) $f(x) = -\sqrt{-x}$

4) $f(x) = -\tan(x)$

5) $f(x) = 5\sqrt{-\frac{1}{3}x}$

6) $f(x) = -3\left(\frac{1}{2}\right)^{-x}$

7) $f(x) = -\frac{1}{2}\cos\left(\frac{1}{4}x\right)$

8) $f(x) = -2\log_5(-4x)$

9) $f(x) = -8\left|\frac{1}{4}x + 2\right|$

10) $f(x) = \log_2(-3x + 5) + 7$

11) $f(x) = -\tan\left(\frac{1}{3}x - 3\right) - 5$

12) $f(x) = -2\sqrt{-\frac{1}{2}x + 2} + 4$

7.4 Inverse Functions

Inverting a function entails reversing the outputs and inputs of the function.

For example, if inputting $x = 1$ into a function f produces an output $f(1) = 3$, then inputting $x = 3$ into the **inverse function** f^{-1} results in the output $f^{-1}(3) = 1$.

Computing Inverse Functions

We can compute inverse functions by switching x and y in the equation for a function, and then solving for y again.

Original function	$f(x) = 2x + 3$
Replace $f(x)$ with y	$y = 2x + 3$
Switch x and y	$x = 2y + 3$
Solve for y	$y = \frac{x-3}{2}$
Replace y with $f^{-1}(x)$	$f^{-1}(x) = \frac{x-3}{2}$

Testing our inverse function on a few sample inputs, we see that it does indeed reverse the outputs and inputs of the original function.

$f(4) = 2(4) + 3 = 11$

$f^{-1}(11) = \frac{11-3}{2} = 4$

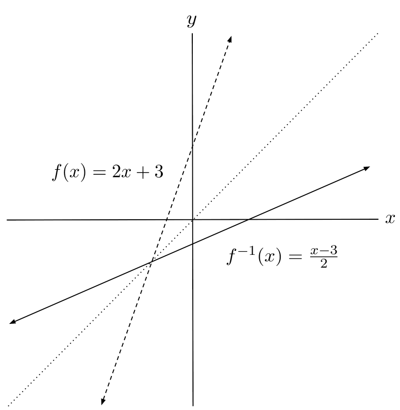
$f(-6) = 2(-6) + 3 = -9$

$f^{-1}(-9) = \frac{-9-3}{2} = -6$

Graphing Inverse Functions

Graphing inverse functions is even easier than computing them: we just have to reflect the original function over the line $y = x$.

This makes sense, intuitively, since computing the inverse function involves switching y and x.

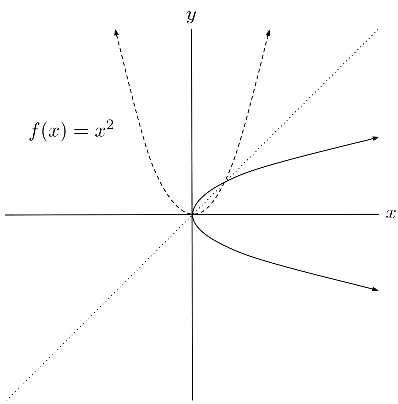


Case when No Inverse Exists

Graphically, we can see that some functions don't have inverse functions. If reflecting the graph over the line $y = x$ causes multiple y-values to be associated with a single x-value, then this breaks the definition of a function, and the resulting graph is not a function.

Algebraically, an inverse function is supposed to take original outputs back to original inputs, but it can't do this if it can't distinguish which input x-value caused the output y-value.

For example, the function $f(x) = x^2$ has $f(2) = f(-2) = 4$, so when a supposed inverse function takes an output of 4, it will not know whether the output came from the input 2 or -2. Therefore, no inverse function can be constructed for $f(x) = x^2$.

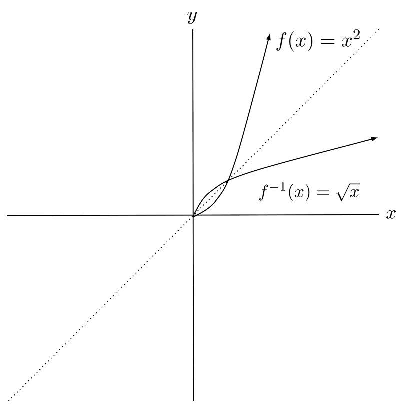


Domain Restrictions

That being said, inverse functions can be created if we restrict the **domain**, the set of allowed inputs.

For example, if we restrict the domain of $f(x) = x^2$ to only positive inputs, then the inverse function would know that an output of 4 comes from an input of 2.

We can also see this graphically -- if we graph $f(x) = x^2$ only for positive values of x, then no x-value has multiple y-values when we reflect the graph over the line $y = x$.



Exercises

Sketch the original function and the graph of the supposed inverse by reflecting the original function f over the line $y = x$. Then, if the inverse function f^{-1} exists, use algebra to find its equation.

1) $f(x) = 2x$

2) $f(x) = 3x - 3$

3) $f(x) = \sqrt{2x}$

4) $f(x) = x^2 + 1$

5) $f(x) = x^3 + 1$

6) $f(x) = |x|$

7) $f(x) = -\frac{2}{x}$

8) $f(x) = \frac{1}{2x-1}$

9) $f(x) = \frac{1}{3x^2}$

10) $f(x) = 3^x + 4$

11) $f(x) = \frac{3}{2}\log_2(x+3)$

12) $f(x) = 4$

13) $f(x) = -3x^2$
restriction: $x \geq 0$

14) $f(x) = \dfrac{1}{2}x^2$
restriction: $x \leq 0$

15) $f(x) = |2x - 3|$
restriction: $x \geq \dfrac{3}{2}$

16) $f(x) = -\left|1 - \dfrac{1}{3}x\right|$
restriction: $x \geq 3$

7.5 Compositions

Compositions of functions consist of multiple functions linked together, where the output of one function becomes the input of another function.

Demonstration

For example, the function $2x^2$ can be thought of as the composition of two functions: the first function squares the input, and then the second function doubles the input.

Using formal notation, we can define the first function that squares the input as $f(x) = x^2$, and the second function that doubles the input as $g(x) = 2x$.

Then the composition can be computed by using the output of f as the input to g. Starting at the end, we can compute the composition by evaluating g in terms of f, and then evaluating f in terms of x.

$$(g \circ f)(x) = g(f(x)) = 2f(x) = 2x^2$$

Or, we can start at the beginning, computing f in terms of x and then evaluating g in terms of the result. Either way, we end up with the same formula for the composition.

$$(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$$

Order of Composition

The order of composition is very important and is not interchangeable.

- The function computed above is $g \circ f$, which applies f first and then g.
- On the other hand, the function $f \circ g$ applies g first and then f, and consequently evaluates to something different: $(f \circ g)(x) = 4x^2$.

Compositions of Many Functions

For compositions of more than two functions, we can compute one step at a time.

Given functions	$f(x) = \sin x$ $g(x) = x^2$ $h(x) = 5x + 1$ $p(x) = \sqrt{x}$
Input f into g	$(g \circ f)(x) = \sin^2 x$
Input $g \circ f$ into h	$(h \circ g \circ f)(x) = 5\sin^2 x + 1$
Input $h \circ g \circ f$ into p	$(p \circ h \circ g \circ f)(x) = \sqrt{5\sin^2 x + 1}$

Exercises

Find the expression for the indicated composition.

1) $(g \circ f)(x) = \_\_\_$
$f(x) = x + 5$
$g(x) = 2x^2$

2) $(g \circ f)(x) = \_\_\_$
$f(x) = 5^x$
$g(x) = |4 - x|$

3) $(h \circ g \circ f)(x) = \_\_\_$
$f(x) = -2^x$
$g(x) = |x + 4|$
$h(x) = \sqrt{x}$

4) $(h \circ g \circ f)(x) = \_\_\_$
$f(x) = 2x$
$g(x) = \dfrac{x}{x-1}$
$h(x) = \sin x$

5) $(p \circ h \circ g \circ f)(x) = \_\_\_$
$f(x) = \sin x$
$g(x) = x^2$
$h(x) = 1 + \sqrt[3]{x}$
$p(x) = \sqrt{x}$

6) $(p \circ h \circ g \circ f)(x) = \_\_\_$
$f(x) = \sqrt{x}$
$g(x) = \tan x$
$h(x) = \log_3 x$
$p(x) = |x|^3$

Solutions

to Exercises

Part 1

Chapter 1.1

1) $x = 4$

2) $x = 1$

3) $x = 2$

4) no solution

5) $x = -3$

6) infinitely many solutions

7) $x = 5$

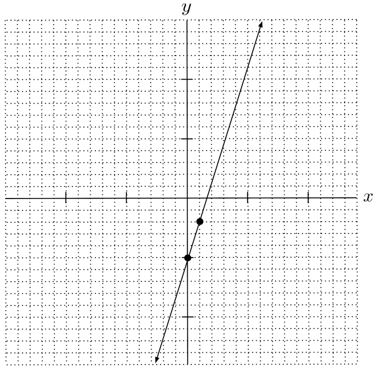
8) infinitely many solutions

9) $x = -6$

10) no solution

Chapter 1.2

1)



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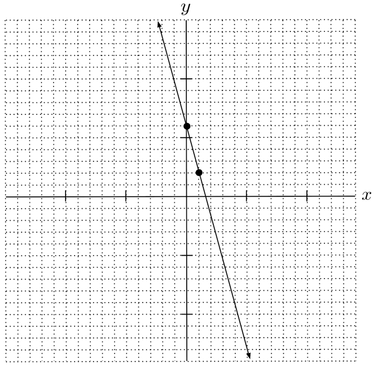

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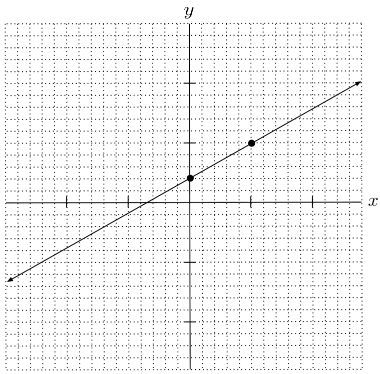

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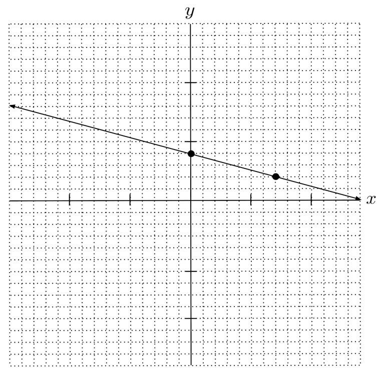

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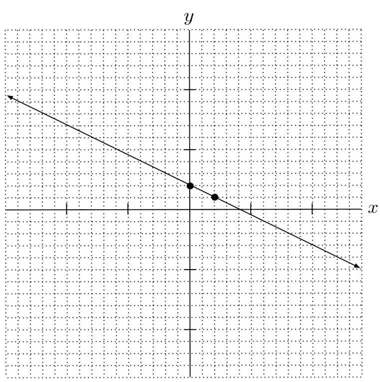

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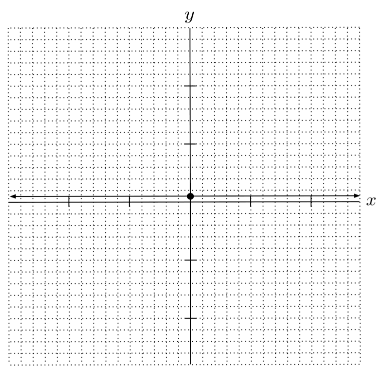

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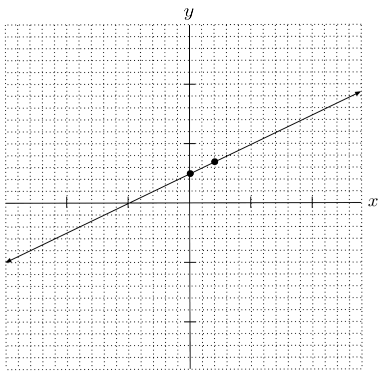

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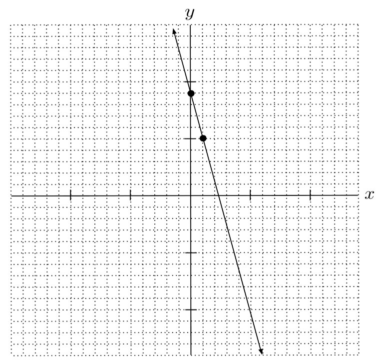


9) $y = -3x + 8$

10) $y = \frac{1}{3}x + 2$

11) $y = \frac{7}{2}x - 1$

12) $y = -\frac{8}{7}x - 6$

13) $y = 3x - 2$

14) $y = -2x + 6$

15) $y = \frac{1}{2}x - 3$

16) $y = -\frac{5}{7}x - \frac{37}{7}$

17) $y = -\frac{3}{2}x + 3$

18) $y = -\frac{1}{4}x + \frac{5}{4}$

19) $y = \frac{6}{5}x - \frac{1}{5}$

20) $y = 5$

Chapter 1.3

1) $y - 5 = 2(x - 1)$

2) $y - 3 = 8(x + 2)$

3) $y+2=\frac{3}{8}\left(x-\frac{1}{2}\right)$

4) $y-\frac{8}{13}=-\frac{12}{5}\left(x+\frac{4}{7}\right)$

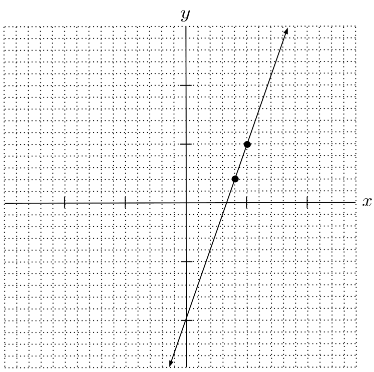
5) $y+1=-2(x-2)$
or $y-1=-2(x-1)$

6) $y-8=3(x-1)$
or $y+7=3(x+4)$

7) $y-3=\dfrac{3}{2}\left(x-\dfrac{1}{3}\right)$
or $y-4=\dfrac{3}{2}(x-1)$

8) $y-\dfrac{1}{2}=\dfrac{1}{5}\left(x+\dfrac{3}{4}\right)$
or $y-\dfrac{3}{4}=\dfrac{1}{5}\left(x-\dfrac{1}{2}\right)$

9)



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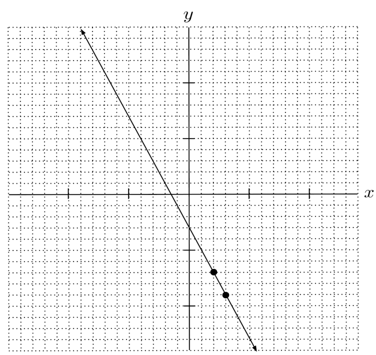

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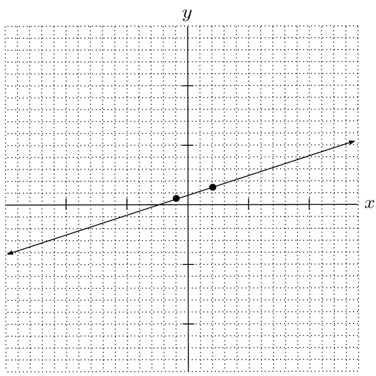

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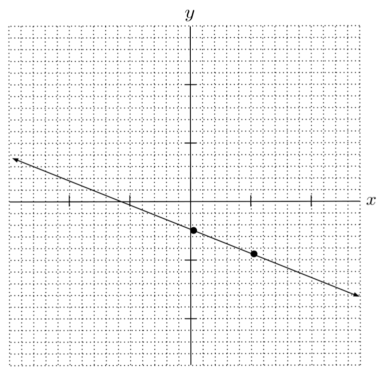


Chapter 1.4

1) $3x - 4y = 4$

2) $2x - 3y = -4$

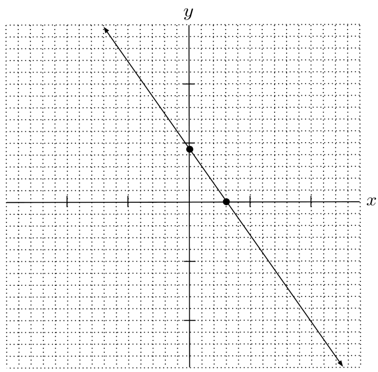
3) $x - 3y = 6$

4) $5x - 4y = 1$

5) $x + y = 2$

6) $12x + 3y = 1$

7)



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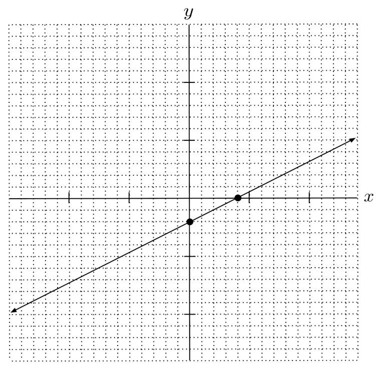

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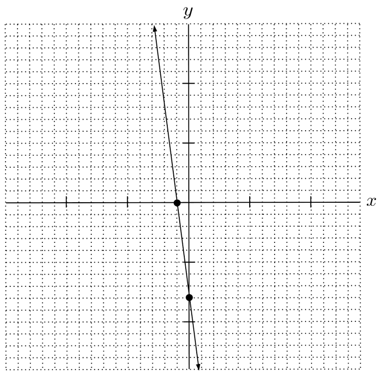

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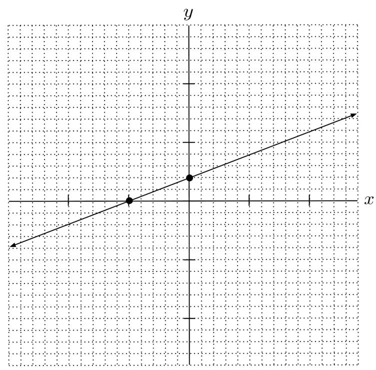


Chapter 1.5

1) $(2, 1)$

2) no solution

3)	$(3, -1)$	4)	$(-1, 7)$
5)	infinitely many solutions	6)	$(1, 1)$
7)	$\left(\frac{11}{10}, \frac{1}{2}\right)$	8)	no solution
9)	$\left(\frac{33}{5}, \frac{8}{5}\right)$	10)	$\left(\frac{35}{23}, \frac{65}{23}\right)$

Part 2

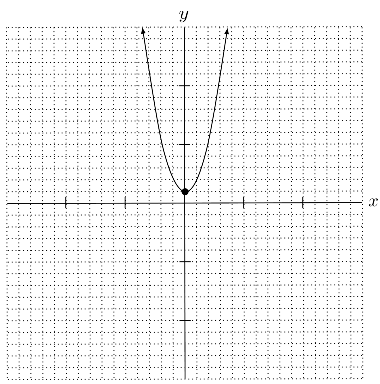
Chapter 2.1

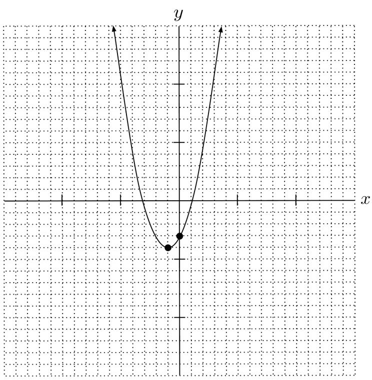
1) $y = x^2 + 1$

up $(0, 1)$ 1

2) $y = x^2 + 2x - 3$

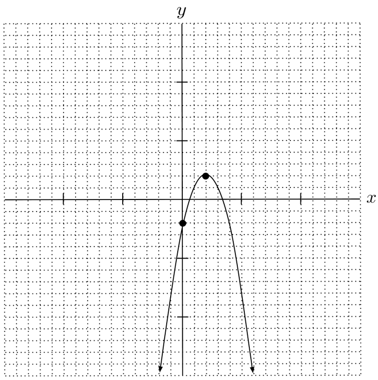
up $(-1, -4)$ -3





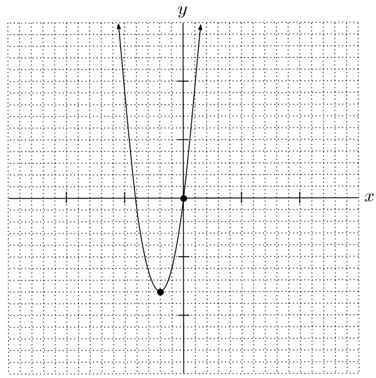
3) $y = -x^2 + 4x - 2$

down $(2, 2)$ -2



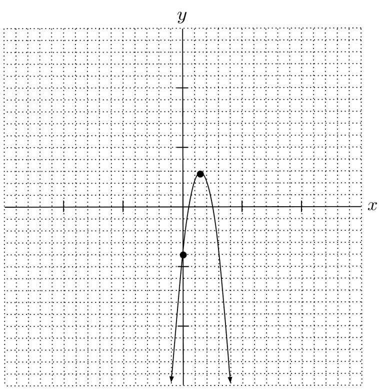
4) $y = 2x^2 + 8x$

up $(-2, -8)$ 0



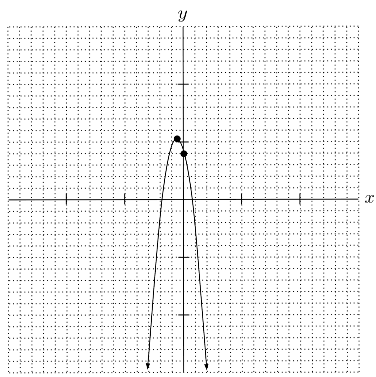
5) $y = -3x^2 + 9x - 4$

down $\left(\frac{3}{2}, \frac{11}{4}\right)$ -4



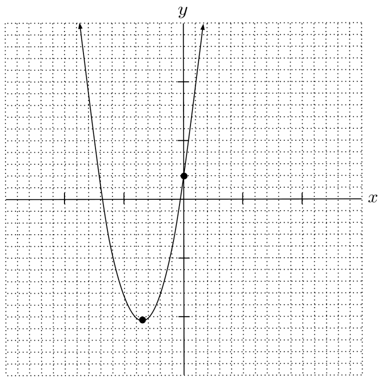
6) $y = -5x^2 - 5x + 4$

down $\left(-\frac{1}{2}, \frac{21}{4}\right)$ 4



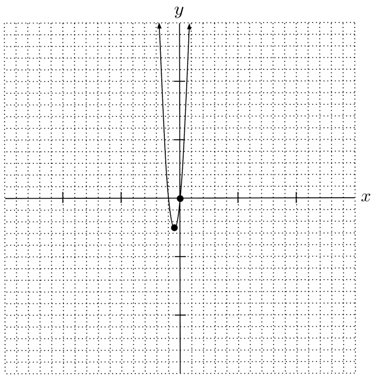
7) $y = x^2 + 7x + 2$

up $\left(-\frac{7}{2}, -\frac{41}{4}\right)$ 2



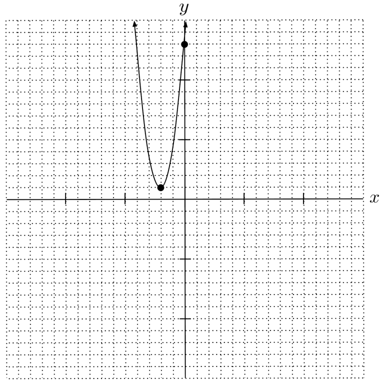
8) $y = 10x^2 + 10x$

up $\left(-\frac{1}{2}, -\frac{5}{2}\right)$ 0



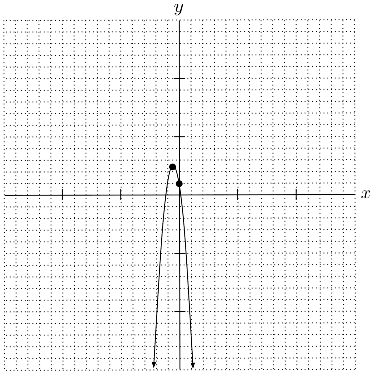
9) $y = 3x^2 + 12x + 13$

up $(-2, 1)$ 13



10) $y = -3x^2 - 4x + 1$

down $\left(-\frac{2}{3}, \frac{7}{3}\right)$ 1



Chapter 2.2

1) $(x+3)(x+4)=0$
$x=-3$ or $x=-4$

2) $(x+7)(x+2)=0$
$x=-7$ or $x=-2$

3) $(x-2)(x-5)=0$
$x=2$ or $x=5$

4) $(x-6)(x-3)=0$
$x=6$ or $x=3$

5) $(x+4)(x-2)=0$
$x=-4$ or $x=2$

6) $(x+7)(x-3)=0$
$x=-7$ or $x=3$

7) $(x+2)(x-5)=0$
$x=-2$ or $x=5$

8) $(x+4)(x-9)=0$
$x=-4$ or $x=9$

9) $4x(x+13)=0$
$x=0$ or $x=-13$

10) $8x(x-8)=0$
$x=0$ or $x=8$

11) $(x+5)(x-5)=0$
$x=-5$ or $x=5$

12) $(x+12)(x-12)=0$
$x=-12$ or $x=12$

13) $(4x+1)(3x+2)=0$
$x=-\dfrac{1}{4}$ or $x=-\dfrac{2}{3}$

14) $(2x-5)(5x-1)=0$
$x=\dfrac{5}{2}$ or $x=\dfrac{1}{5}$

15) $(3x+4)(2x-1)=0$
$x=-\dfrac{4}{3}$ or $x=\dfrac{1}{2}$

16) $(7x-2)(3x+5)=0$
$x=\dfrac{2}{7}$ or $x=-\dfrac{5}{3}$

Chapter 2.3

1) $x=\frac{-3\pm\sqrt{37}}{2}$

2) $x=\frac{3}{2}$

3) $x = -1$ or $x = 3$

4) no solution

5) $x = -\frac{2}{5}$

6) $\frac{1 \pm \sqrt{10}}{3}$

7) $x = \frac{7}{3}$

8) no solution

9) $\frac{3 \pm \sqrt{29}}{10}$

10) $x = \frac{7}{11}$

Chapter 2.4

1) $x = -\frac{3}{2} \pm \sqrt{\frac{13}{4}}$

2) $x = \frac{1}{2} \pm \sqrt{\frac{9}{4}}$

3) $x = 1 \pm \sqrt{4}$

4) $x = -2 \pm \sqrt{11}$

5) $x = -\frac{7}{2} \pm \sqrt{\frac{53}{4}}$

6) no solution

7) $x = \frac{1}{3} \pm \sqrt{\frac{10}{9}}$

8) $x = -\frac{3}{10} \pm \sqrt{\frac{29}{100}}$

9) no solution

10) $x = -\frac{1}{14} \pm \sqrt{\frac{85}{196}}$

Chapter 2.5

1) $y = (x+1)^2 + 2$
$(-1, 2)$ up

2) $y = (x-3)^2 - 5$
$(3, -5)$ up

3) $y = 2(x+5)^2 - 55$
$(-5, -55)$ up

4) $y = -3(x-1)^2 + 4$
$(1, 4)$ down

5) $y = 2\left(x - \frac{1}{2}\right)^2 - \frac{3}{2}$
$\left(\frac{1}{2}, -\frac{3}{2}\right)$ up

6) $y = -\frac{1}{3}(x+3)^2 - 7$
$(-3, -7)$ down

7) $y = \frac{3}{4}\left(x - \frac{4}{3}\right)^2 + \frac{5}{3}$
$\left(\frac{4}{3}, \frac{5}{3}\right)$ up

8) $y = -\frac{2}{3}(x+6)^2 + 7$
$(-6, 7)$ down

Chapter 2.6

1) $(2, 16)$ $(3, 24)$

2) $(-5, 23)$

3) $(1, 19)$ $(-1, 5)$

4) no solution

5) $\left(\frac{2}{3}, \frac{34}{9}\right)$

6) $(-2, 0)$ $(-5, 0)$

7) no solution

8) $\left(\frac{5}{2}, -\frac{35}{4}\right)$ $(3, -11)$

Part 3

Chapter 3.1

1) $x \le 2 \quad (-\infty, 2]$

2) $x > -5 \quad (-5, \infty)$

3) $x > -3 \quad (-3, \infty)$

4) $x < 7 \quad (-\infty, 7)$

5) $x \ge \frac{3}{2} \quad \left[\frac{3}{2}, \infty\right)$

6) $x \le -\frac{1}{3} \quad \left(-\infty, -\frac{1}{3}\right]$

7) $x < 9 \quad (-\infty, 9)$

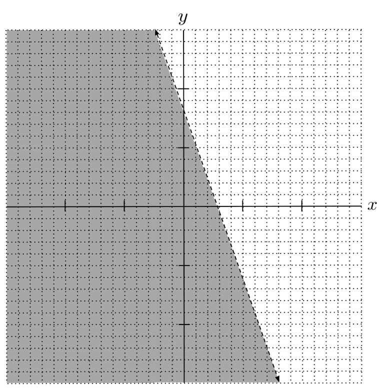
8) $x > -\frac{12}{7} \quad \left(-\frac{12}{7}, \infty\right)$

9) $x \ge -\frac{22}{3} \quad \left[-\frac{22}{3}, \infty\right)$

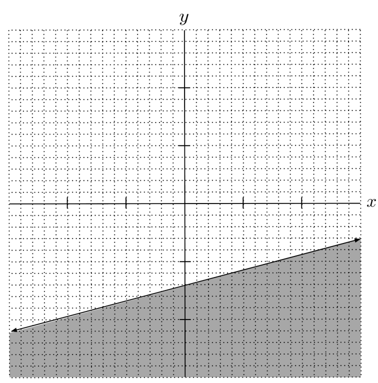
10) $x < \frac{4}{5} \quad \left(-\infty, \frac{4}{5}\right)$

Chapter 3.2

1)



2)



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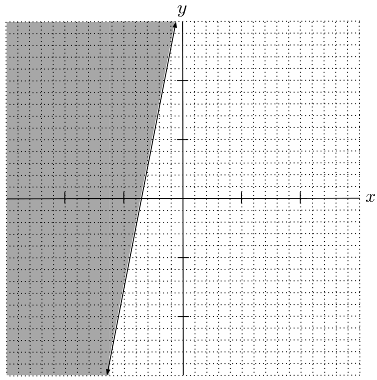

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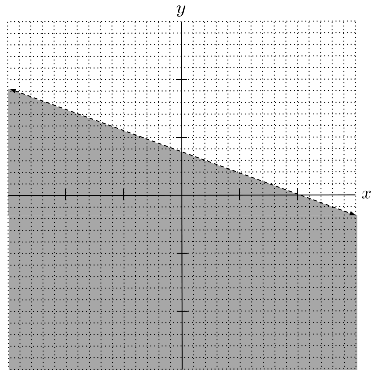

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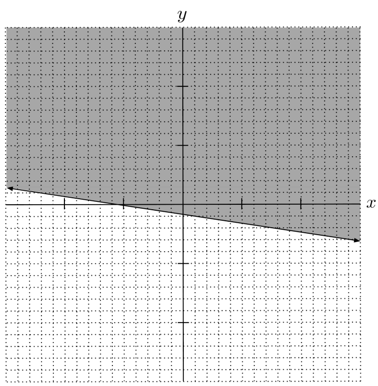

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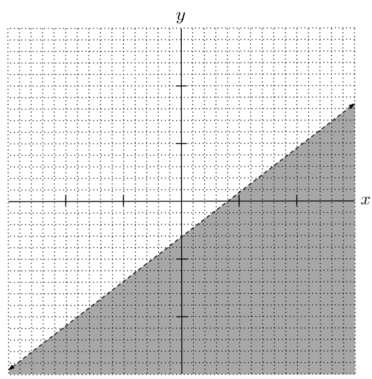

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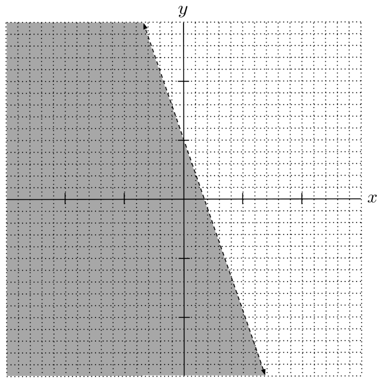

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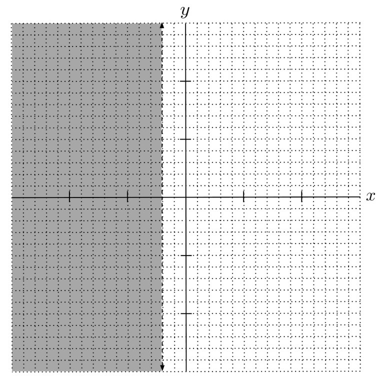
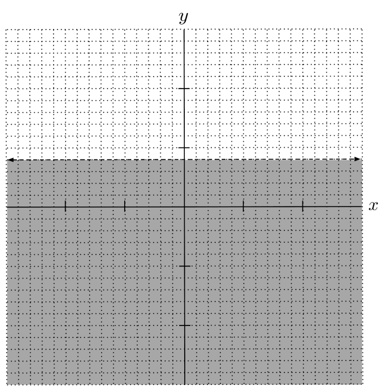
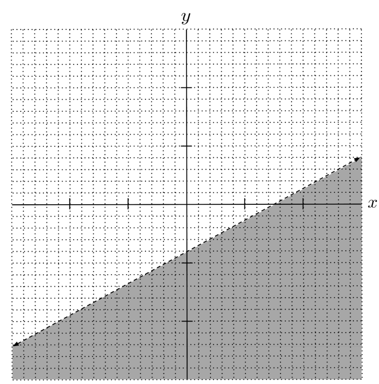


9)



10)



Chapter 3.3

1) $(-\infty, -2] \cup [2, \infty)$

2) $(-\infty, \infty)$

3) $(-\infty, -3) \cup (3, \infty)$

4) $(2, 7)$

5) $\emptyset$

6) $\left(-\infty, -\frac{3}{2}\right] \cup \left[5, \infty\right)$

7) $\left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{1}{2}, \infty\right)$

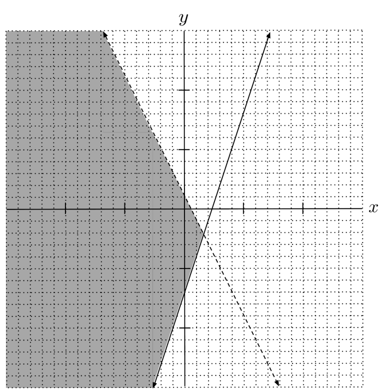
8) $(-10, 3)$

9) $\left[-\frac{1}{5}, \frac{7}{2}\right]$

10) $(-\infty, \infty)$

Chapter 3.4

1)



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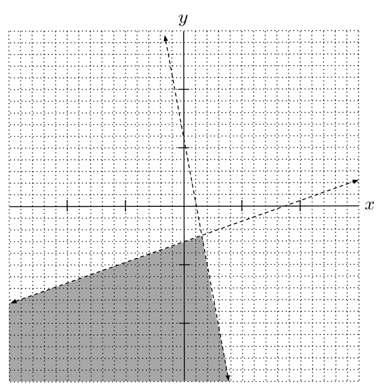
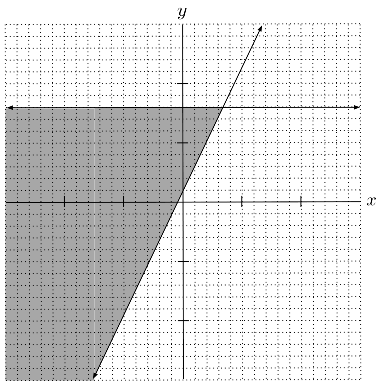


3)



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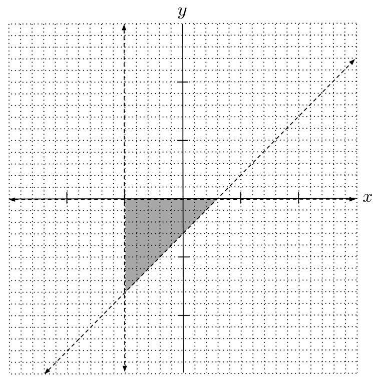
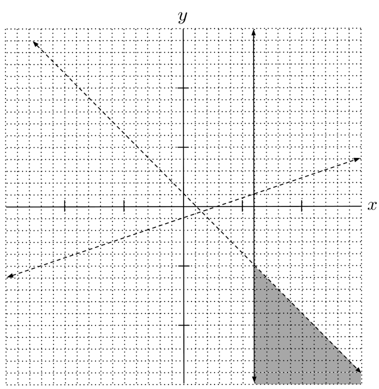
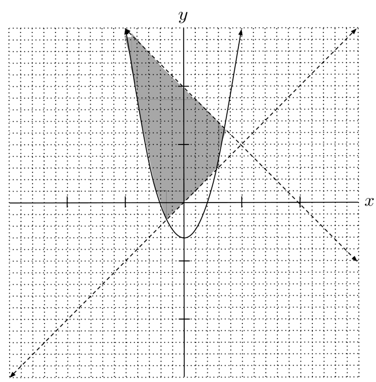


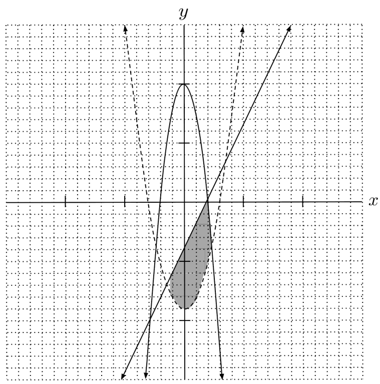
5)



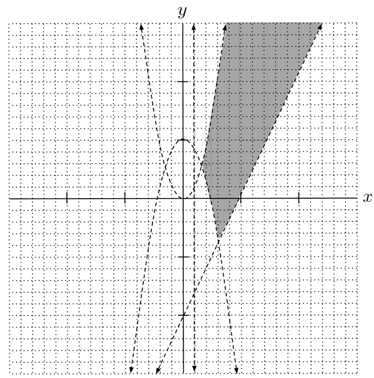
6)



7)



8)



Part 4

Chapter 4.1

1) $p(x) = 3x^4 + 8x^3 - 7x$
$p(x) \to +\infty$ as $x \to +\infty$
$p(x) \to +\infty$ as $x \to -\infty$

2) $p(x) = x^3 - 2x + 1$
$p(x) \to +\infty$ as $x \to +\infty$
$p(x) \to -\infty$ as $x \to -\infty$

3) $p(x) = -x^6 - 9x^4 + 3x^3 + 2$
$p(x) \to -\infty$ as $x \to +\infty$
$p(x) \to -\infty$ as $x \to -\infty$

4) $p(x) = -10x^9 + 5x^4 - x^3 - x^2$
$p(x) \to -\infty$ as $x \to +\infty$
$p(x) \to +\infty$ as $x \to -\infty$

5) $p(x) = -4x^5 + 6x^3 - 2x + 1$
$p(x) \to -\infty$ as $x \to +\infty$
$p(x) \to +\infty$ as $x \to -\infty$

6) $p(x) = x^4 + 4x^3 - 5x + 1$
$p(x) \to +\infty$ as $x \to +\infty$
$p(x) \to +\infty$ as $x \to -\infty$

7) $p(x) = -5x^6 - 6x^5 + x^4 + 8x - 1$
$p(x) \to -\infty$ as $x \to +\infty$
$p(x) \to -\infty$ as $x \to -\infty$

8) $p(x) = x^{11} - 3x^8 - x^4 + 2$
$p(x) \to +\infty$ as $x \to +\infty$
$p(x) \to -\infty$ as $x \to -\infty$

9) $p(x) = x^3 + x^2 + x + 1$
$p(x) \to +\infty$ as $x \to +\infty$
$p(x) \to -\infty$ as $x \to -\infty$

10) $p(x) = -x^3 + 3x^2 + x - 3$
$p(x) \to -\infty$ as $x \to +\infty$
$p(x) \to +\infty$ as $x \to -\infty$

11) $p(x) = 2x^8 - 4x^5 + x^3 - 2$
$p(x) \to +\infty$ as $x \to +\infty$
$p(x) \to +\infty$ as $x \to -\infty$

12) $p(x) = -2x^7 - 4x^6 + 3x^3 + 6x^2$
$p(x) \to -\infty$ as $x \to +\infty$
$p(x) \to +\infty$ as $x \to -\infty$

Chapter 4.2

1) $\frac{1}{2}$

2) $-\frac{7}{2}, \frac{9}{2}$

3) $-\frac{2}{3}, 4$

4) $-\frac{1}{7}, \frac{6}{7}$

5) 5

6) $-3, 4$

7) $-4, \frac{11}{5}$

8) $-5, 7$

Chapter 4.3

1) $3(x+1)(x+2)(x+3)$
zeros: $-1, -2, -3$

2) $(2x-1)(x+1)(x-3)$
zeros: $\dfrac{1}{2}, -1, 3$

3) $(x+1)(x-1)(x-2)^2$
zeros: $\pm 1, 2$

4) $(x^2+3)(x-1)(x-2)$
zeros: $1, 2$

5) $(x-1)^2(x+\sqrt{2})(x-\sqrt{2})$
zeros: $1, \pm\sqrt{2}$

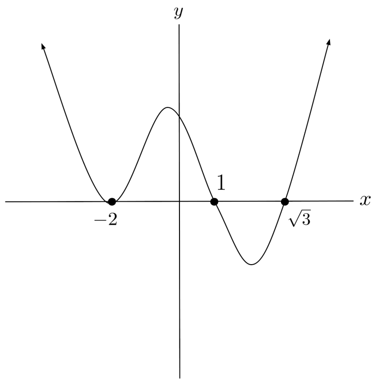
6) $(2x-1)(x+2)(x+1)^2$
zeros: $\dfrac{1}{2}, -2, -1$

7) $2(x-1)(x-3)(x^2+2)$
zeros: $1, 3$

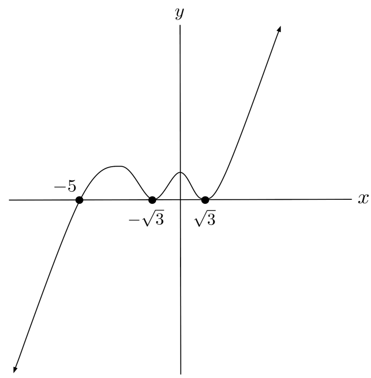
8) $(x+2)(x-2)(7x+3)(3x^2+x+1)$
zeros: $\pm 2, -\dfrac{3}{7}$

Chapter 4.4

1)



2)



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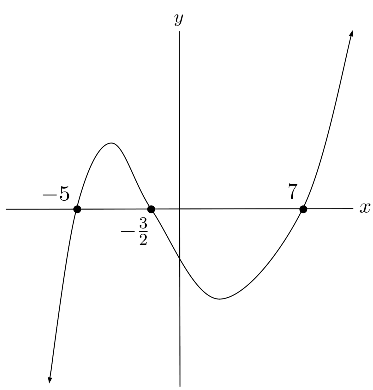
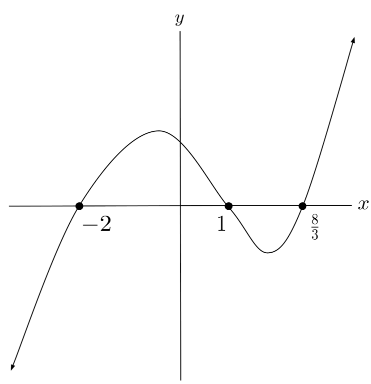
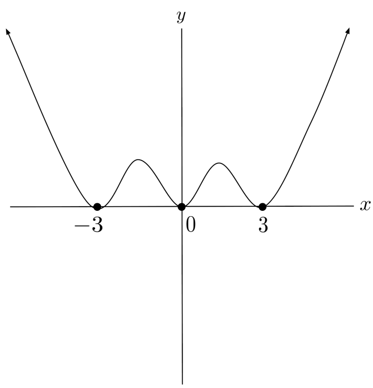


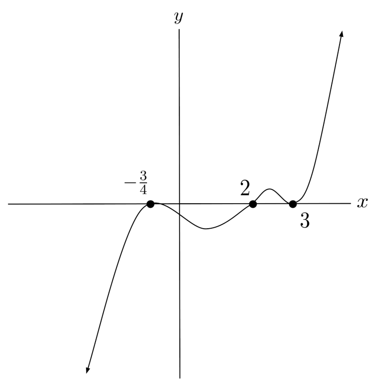
4)



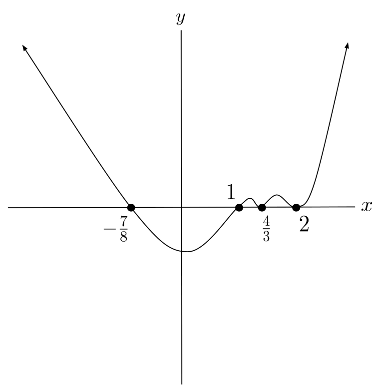
5)


y
x
−3
0
3

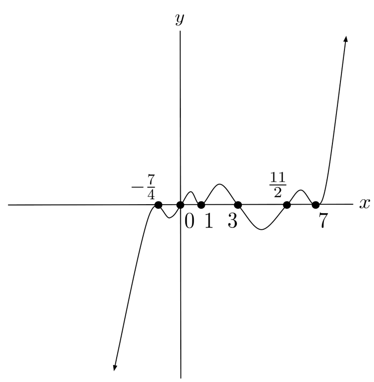
6)


y
x
$-\frac{3}{4}$
2
3

7)


y
x
$-\frac{7}{8}$
1
$\frac{4}{3}$
2

8)


y
x
$-\frac{7}{4}$
0
1
3
$\frac{11}{2}$
7

Part 5

Chapter 5.1

1) $r(x) = 2x^3 + \frac{7}{4} + \frac{3}{2x^2}$

2) $r(x) = -3x^5 + x^3 - \frac{2}{3}x + \frac{x^3+1}{3x^4}$

3) $r(x) = x - 1 + \frac{4}{x+3}$

4) $r(x) = 3 - \frac{7}{x+1}$

5) $r(x) = 2x^4 - x^3 + 2x^2 - 3x + 5 - \frac{5}{x+2}$

6) $r(x) = -2x^3 - x^2 + 2 - \frac{2}{2x-1}$

7) $r(x) = 3x^2 + 5 + \frac{2x-3}{x^2+5}$

8) $r(x) = -3x^4 + x^3 - 2x^2 + \frac{x^2+2}{x^3+2x-1}$

9) $r(x) = x^2 - 2x - 1 + \frac{x^3-3x-2}{x^5+x+1}$

10) $r(x) = 2x^5 + x^2 - 3 + \frac{1}{3x^3+2x^2}$

11) $r(x) = x^7 + 6x^3 - 2x + 7 + \frac{x^3+2x}{x^6-4x^3+2}$

12) $r(x) = -2x^8 + 3x^5 + x^4 - 3x + 2 + \frac{x^3+3x^2+x-2}{9x^4-x^3+x+1}$

Chapter 5.2

1) $y = 5$

2) $y = -2$

3) none

4) $y = 0$

5) $y = \frac{1}{2}$

6) $y = -\frac{3}{2}$

7) $y = 0$

8) none

9) $y = \frac{8}{9}$

10) $y = -2$

Chapter 5.3

1) $x = 4$

2) $x = -\frac{5}{3}$

3) none

4) $x = -1, -2$

5) $x = 3$

6) $x = -4, -3, 3$

7) $x = 1, 2$

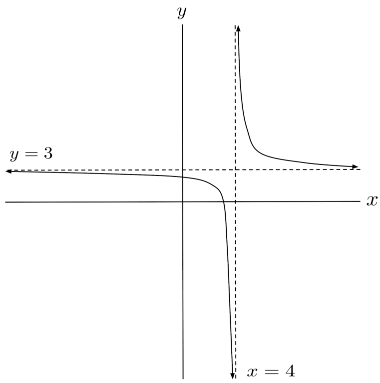
8) $x = -1, 4$

9) none

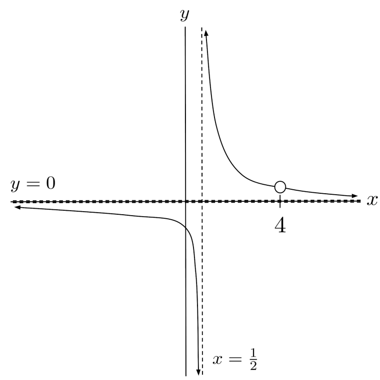
10) $x = -\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$

Chapter 5.4

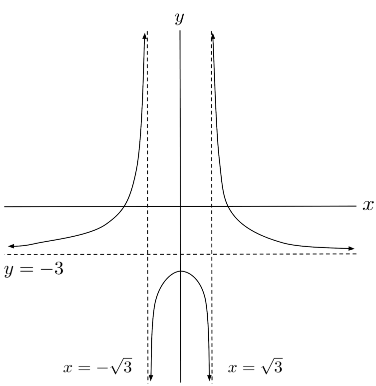
1)



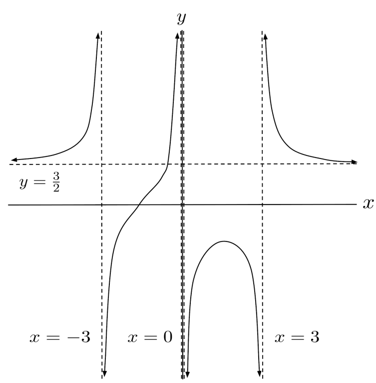
2)



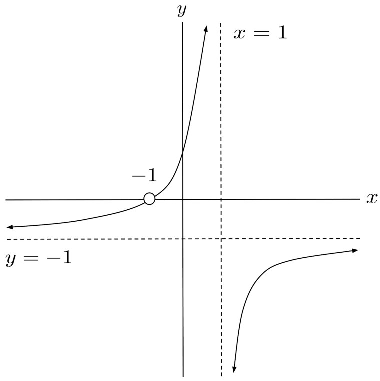
3)



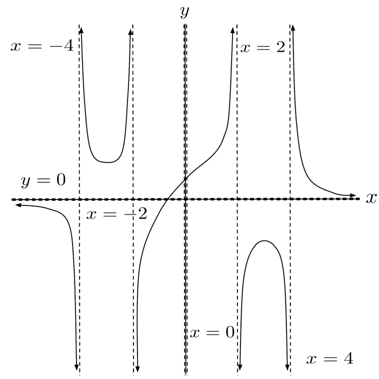
4)



5)

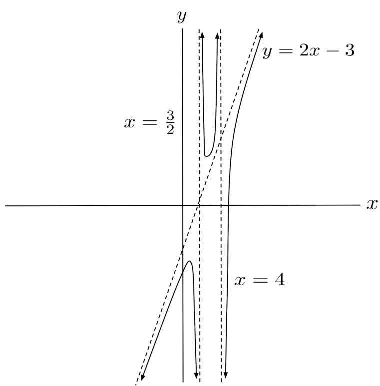


6)

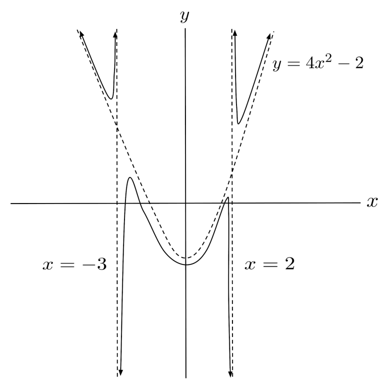


Chapter 5.5

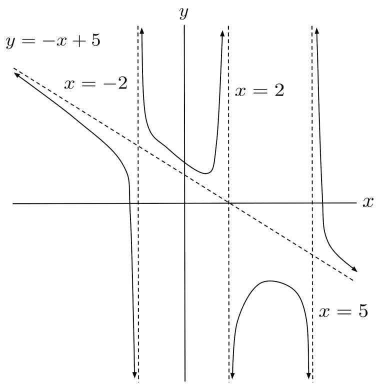
1)



2)

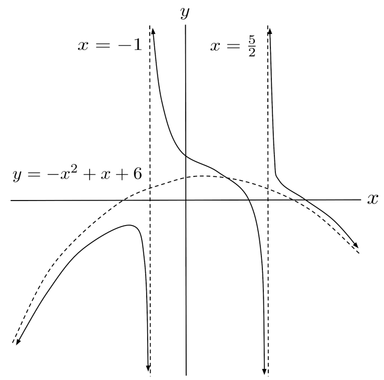


3)



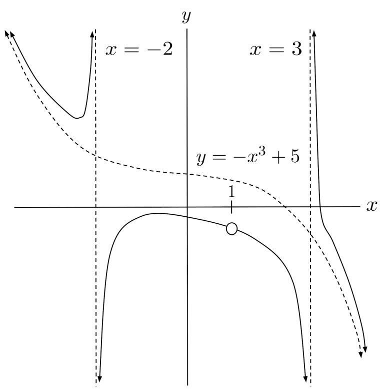
y
$y = -x + 5$
$x = -2$
$x = 2$
x
$x = 5$

4)



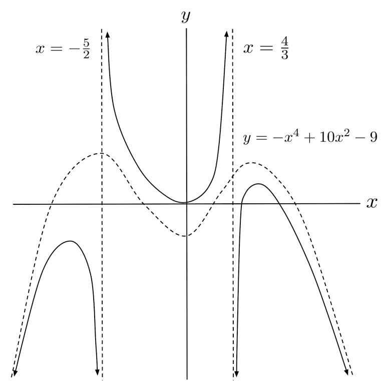
y
$x = -1$
$x = \frac{5}{2}$
$y = -x^2 + x + 6$
x

5)



y
$x = -2$
$x = 3$
$y = -x^3 + 5$
1
x

6)

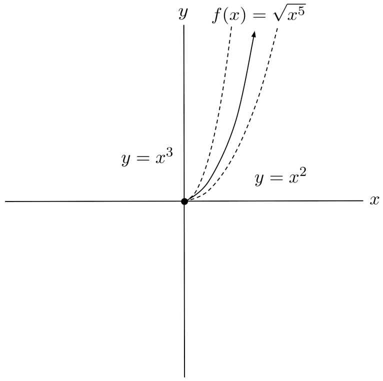


y
$x = -\frac{5}{2}$
$x = \frac{4}{3}$
$y = -x^4 + 10x^2 - 9$
x

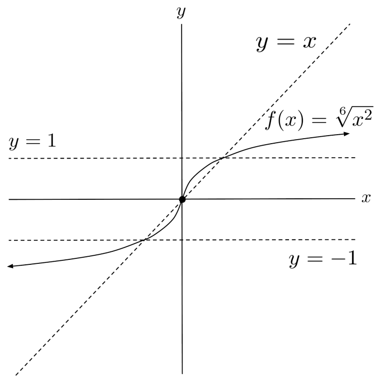
Part 6

Chapter 6.1

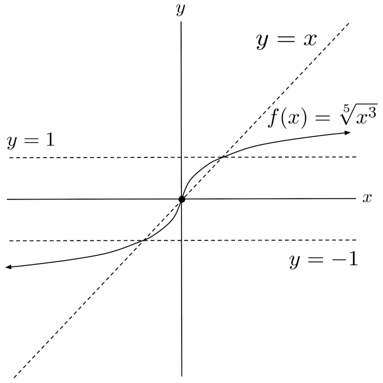
1)



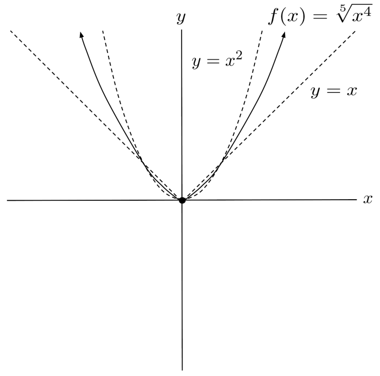
2)



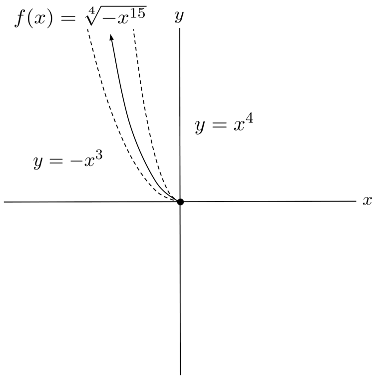
3)



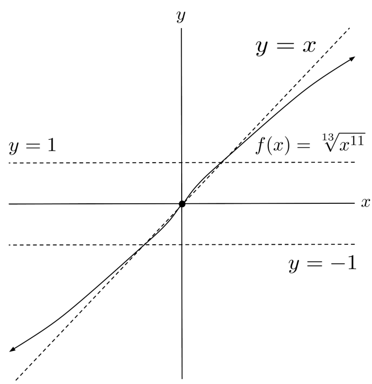
4)



5)



6)



7) $x = 1$

8) $x = -27$

9) no solution

10) $x = \frac{3-\sqrt{5}}{2}$

11) $x = -\frac{5}{2}$

12) $x = 6$

13) no solution

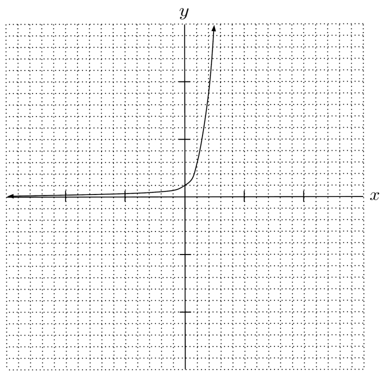
14) $x = -2$

15) $x = \frac{3\pm\sqrt{6}}{3}$

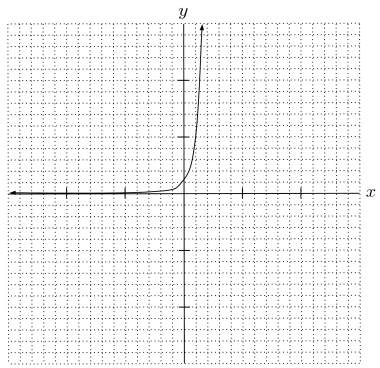
16) $x = 6 - 2\sqrt{11}$

Chapter 6.2

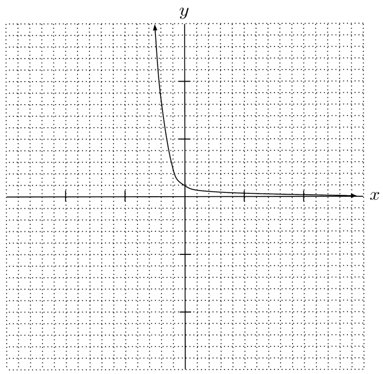
1)



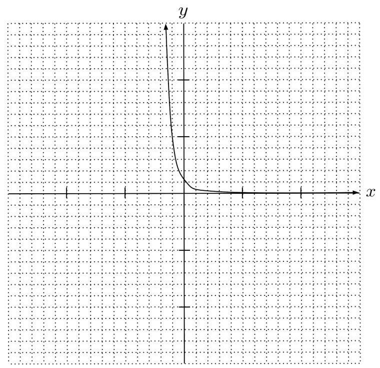
2)



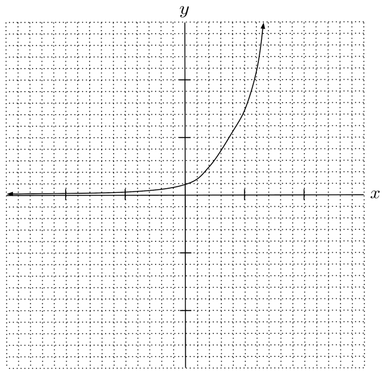
3)



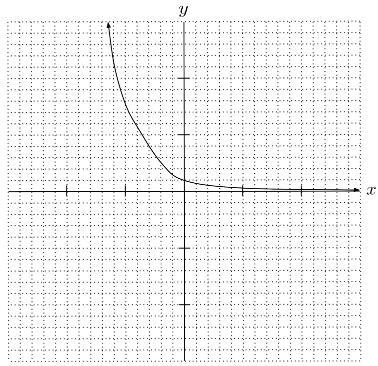
4)



5)



6)



7) $\log_3 10 \approx 2.096$

8) $\log_5 7 \approx 1.209$

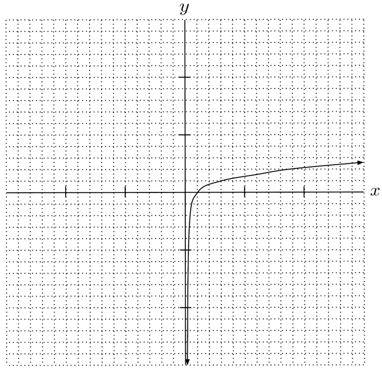
9) $\log_{\frac{1}{3}} \frac{1}{10} \approx 2.096$

10) $\log_{\frac{1}{5}} \frac{1}{2} \approx 0.431$

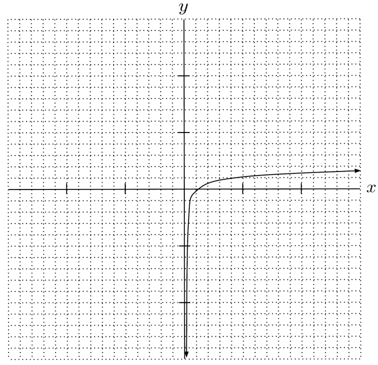
11) $\log_{\frac{3}{2}} 9 \approx 5.419$

12) $\log_{\frac{2}{3}} \frac{1}{5} \approx 3.969$

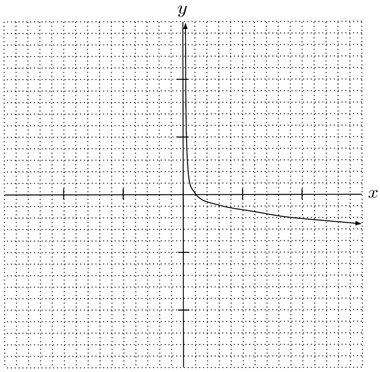
13) $f(x) = \log_3 x$



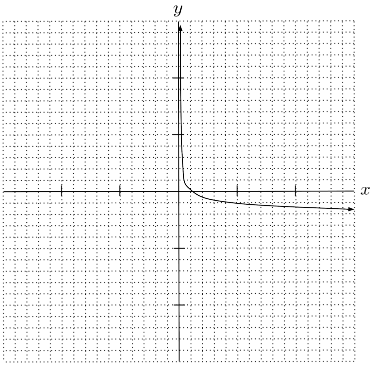
14) $f(x) = \log_5 x$



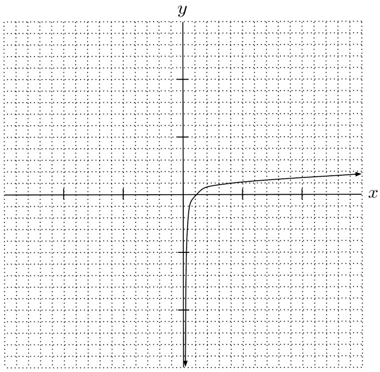
15) $f(x) = \log_{\frac{1}{3}} x$



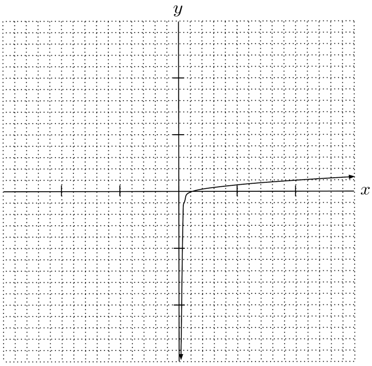
16) $f(x) = \log_{\frac{1}{5}} x$



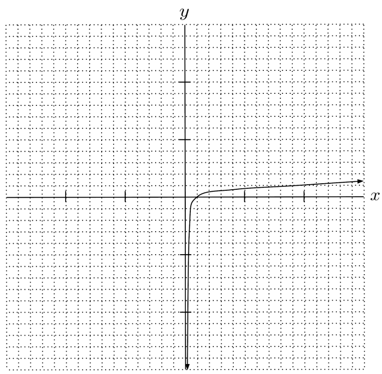
17) $f(x) = \log_4 x$



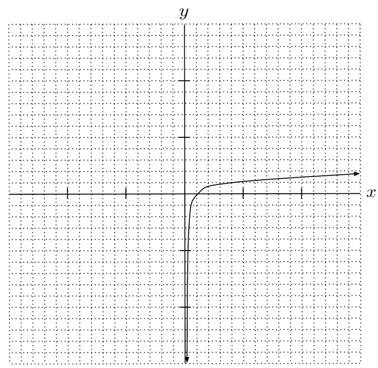
18) $f(x) = \log_{10} x$



19) $f(x) = \log_9 x$



20) $f(x) = \log_4 x$



Chapter 6.3

1) $x = -1, 7$

2) $x = \frac{4}{5}, -2$

3) no solution

4) $x = \pm 1$

5) $x = 2, -4$

6) $x = \frac{2}{3}, 2$

7) $x = 0, \frac{1}{2}$

8) $x = 1, -\frac{5}{4}, \frac{1 \pm \sqrt{17}}{8}$

9) $x = -\frac{1}{2}$

10) $x = 0, \frac{2}{3}$

11) $x = \pm 2$

12) $x = \pm\, 2, 1 - \sqrt{3}$
$- 1 + \sqrt{3}$

13) $x = -2, 0$

14) $x = \sqrt{\frac{1}{2}}, \frac{1-\sqrt{3}}{2}$

Chapter 6.4

1) $a \approx 6.58$
$b \approx 2.39$
$B = 20°$

2) $a \approx 5.96$
$c \approx 7.78$
$A = 50°$

3) $c = \sqrt{5} \approx 2.24$
$A \approx 63.43°$
$B \approx 26.57°$

4) $b = 6$
$A \approx 53.13°$
$B \approx 36.87°$

5) 1

6) -1

7) $\frac{\sqrt{3}}{2}$

8) $\frac{\sqrt{3}}{3}$

9) $-\frac{\sqrt{2}}{2}$

10) undefined

11) $\frac{\sqrt{2}}{2}$

12) $-\frac{1}{2}$

13) 0

14) $-\frac{1}{2}$

15) 1

16) $\frac{\sqrt{2}}{2}$

17) $-\frac{1}{2}$

18) $\frac{1}{2}$

19) $\frac{1}{2}$

20) $-\sqrt{3}$

21) -1

22) $\frac{\sqrt{3}}{3}$

23) 0

24) -1

25) 1

26) 0

27) 1

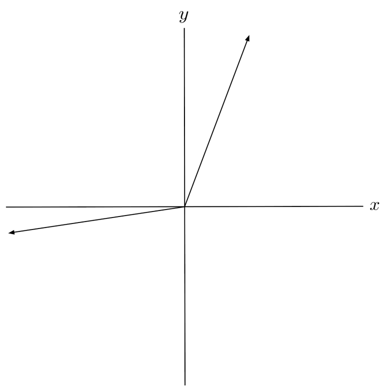
28) $\frac{\sqrt{3}}{2}$

29) $-\sqrt{3}$

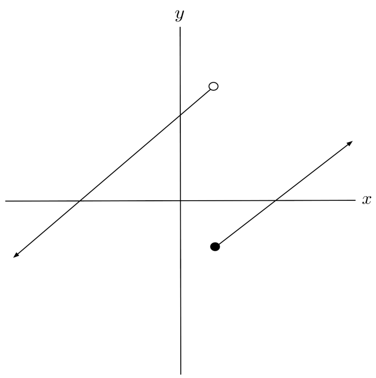
30) -1

Chapter 6.5

1)



2)



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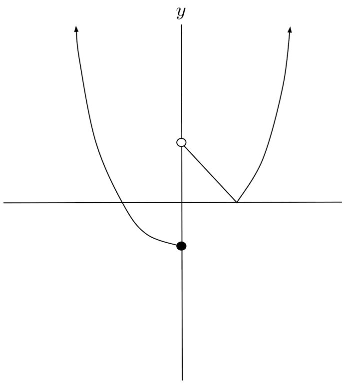

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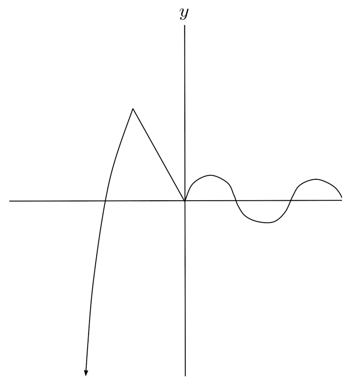

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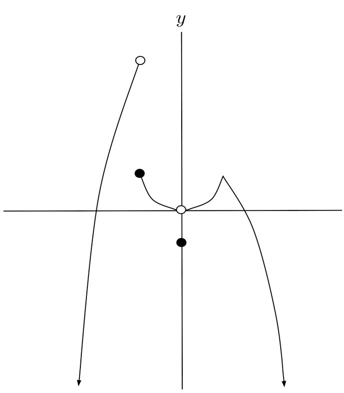
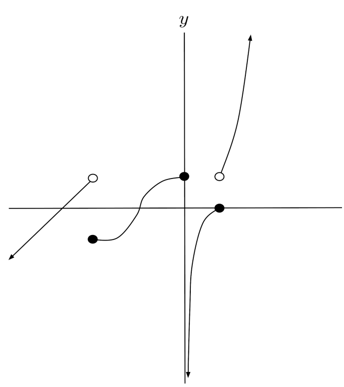


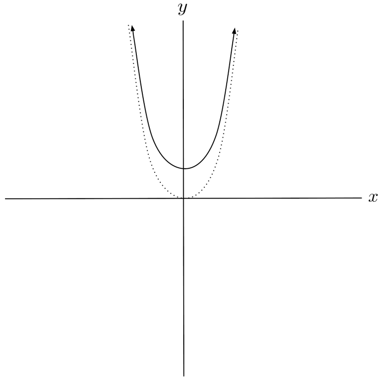
6)



Part 7

Chapter 7.1

1)



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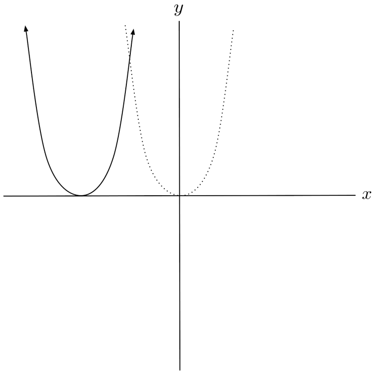
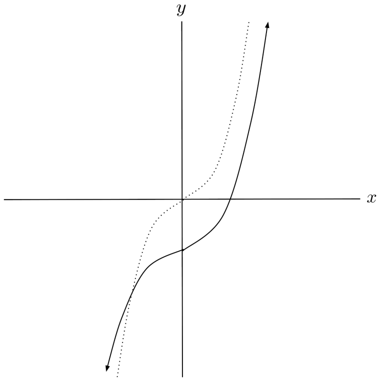


3)



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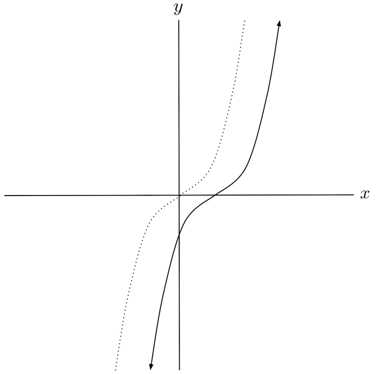

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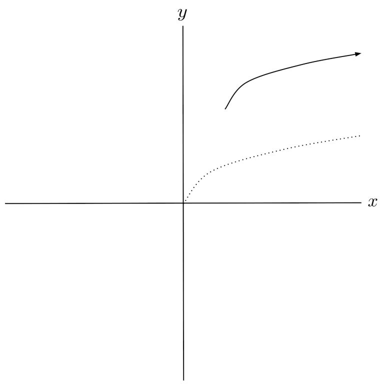

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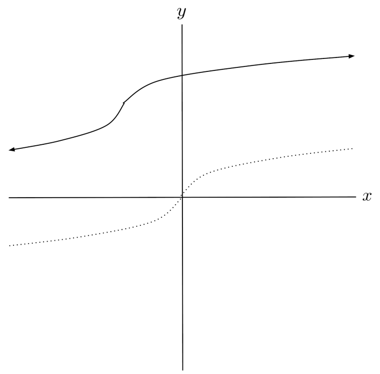

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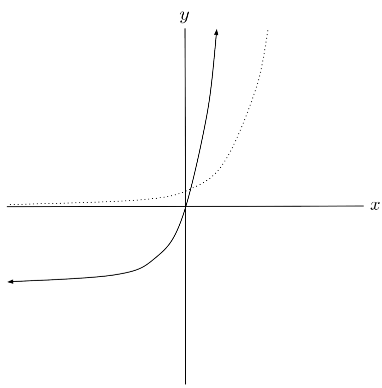

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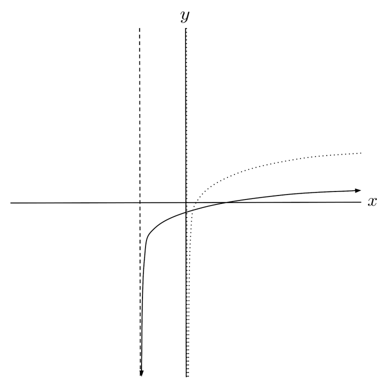

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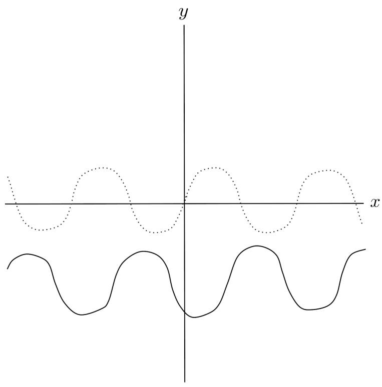

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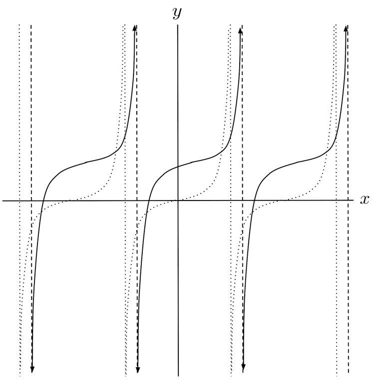

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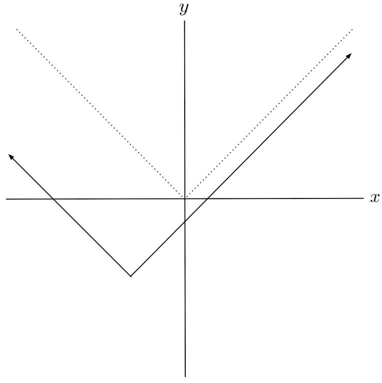

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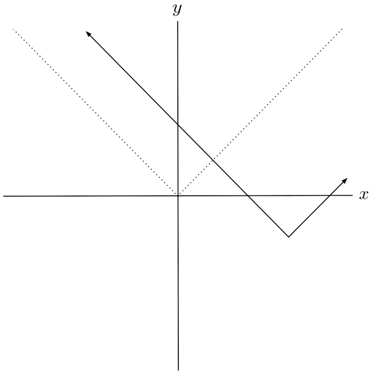
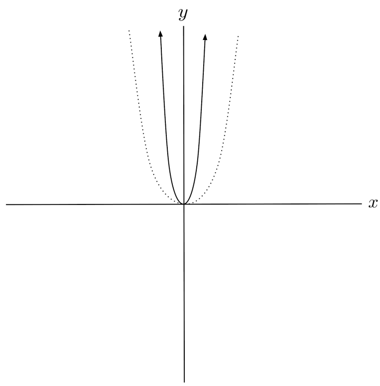
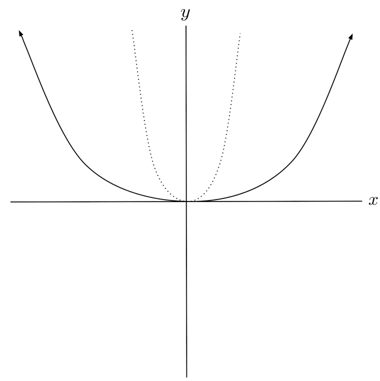


Chapter 7.2

1)



2)



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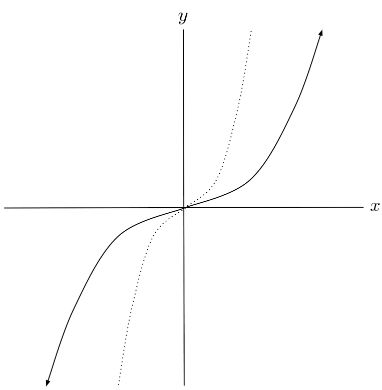
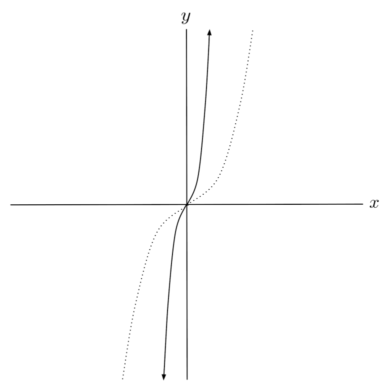


4)



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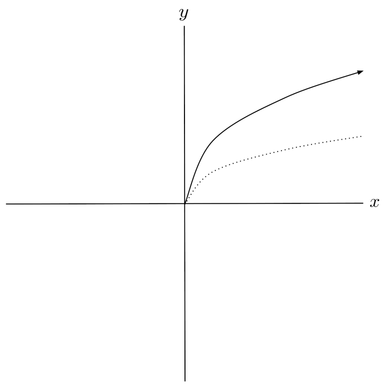

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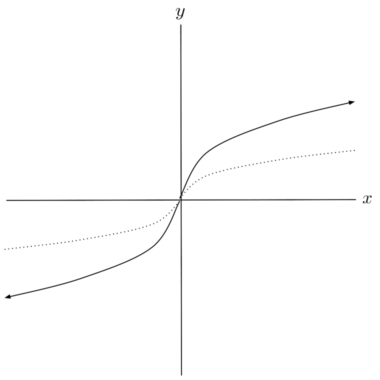

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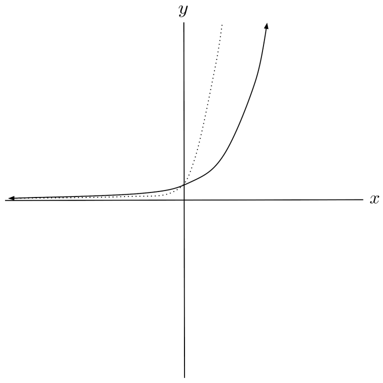

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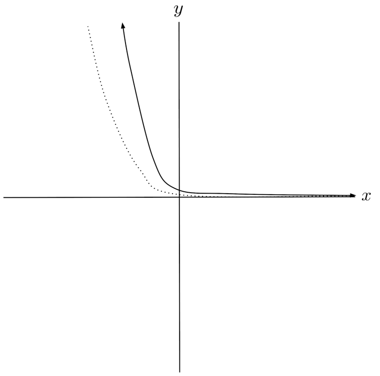

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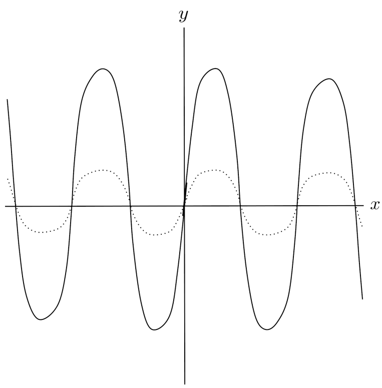

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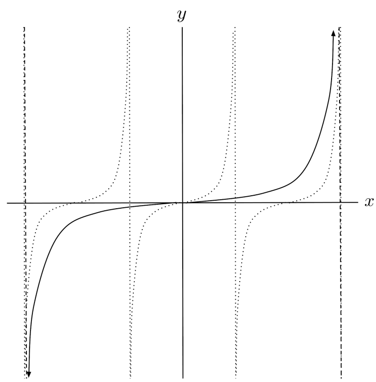

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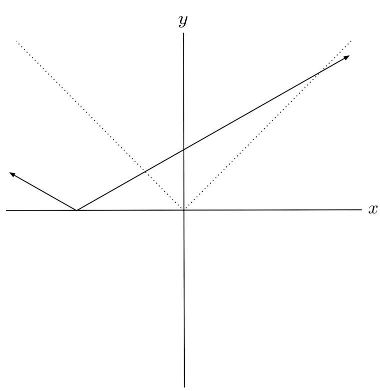

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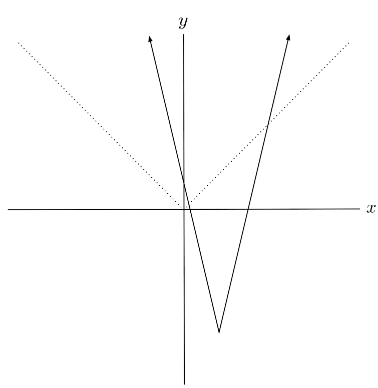

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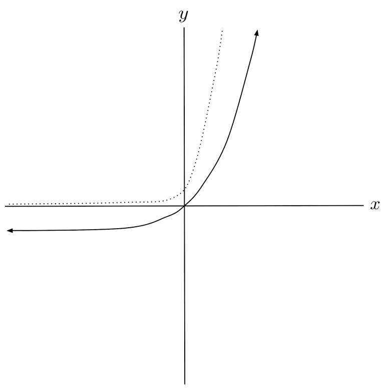

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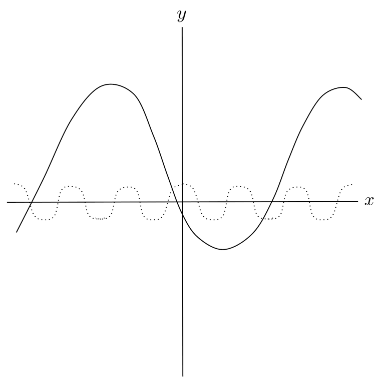

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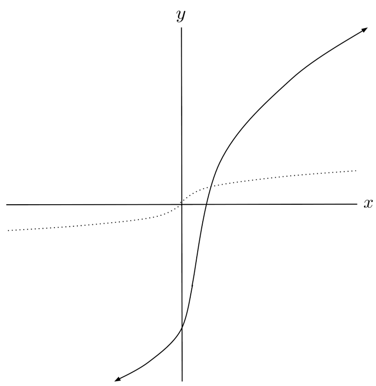

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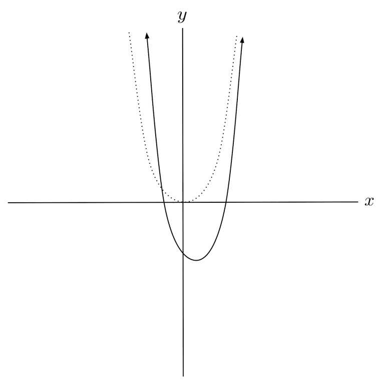
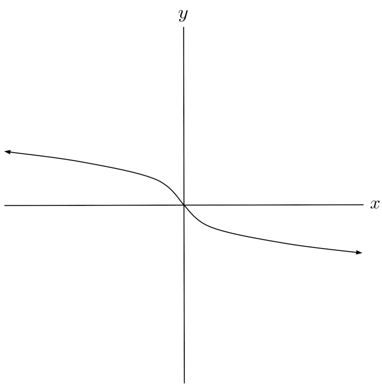
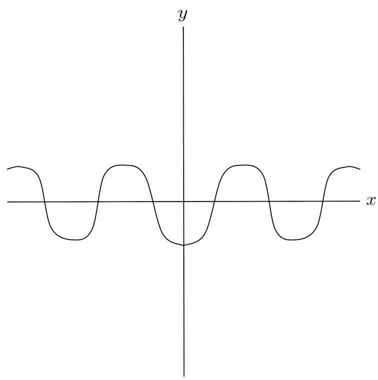


Chapter 7.3

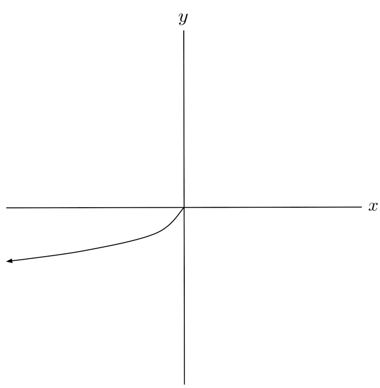
1)



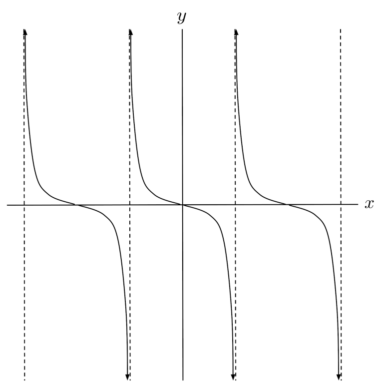
2)



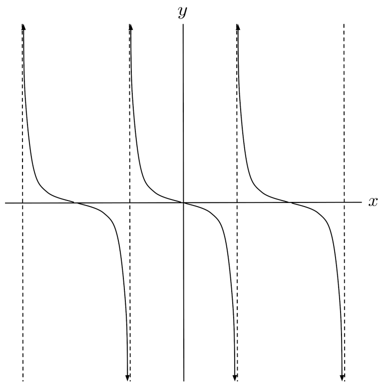
3)



4)



5)



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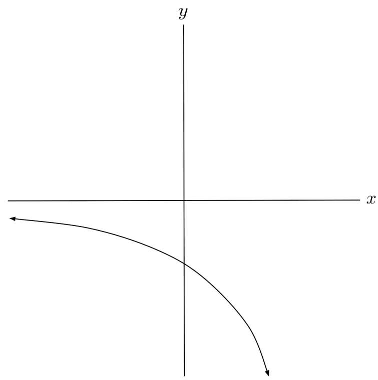

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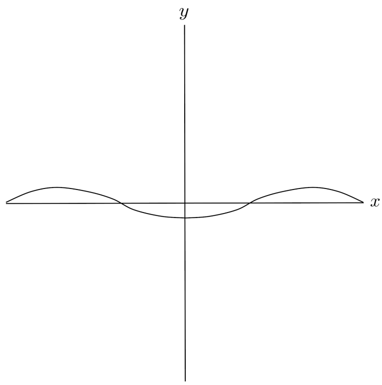

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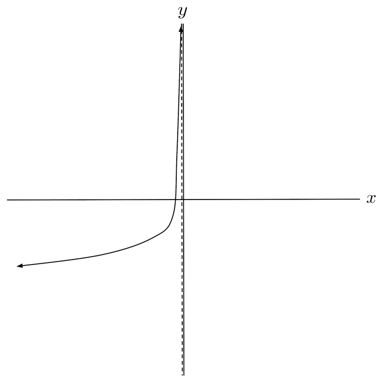

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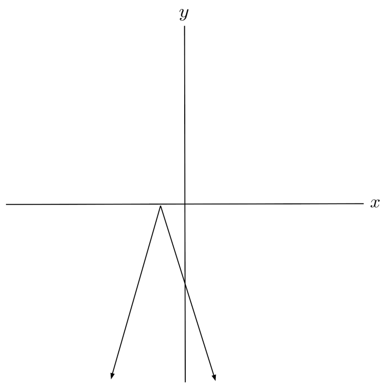

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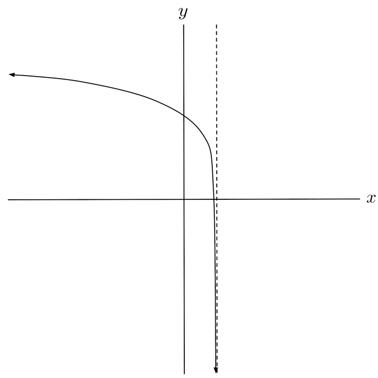

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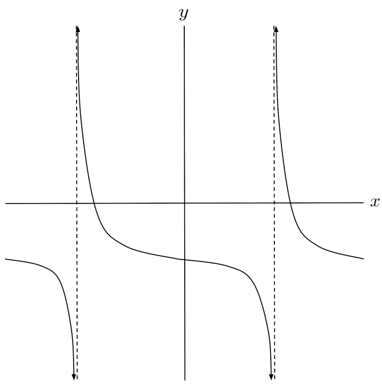
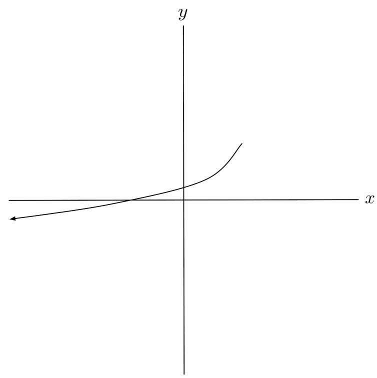
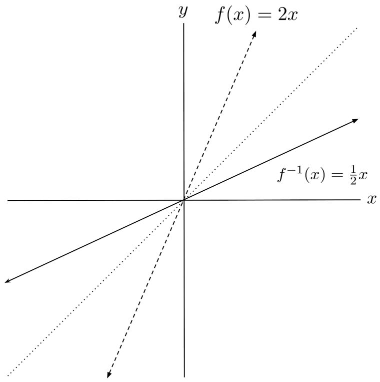


12)

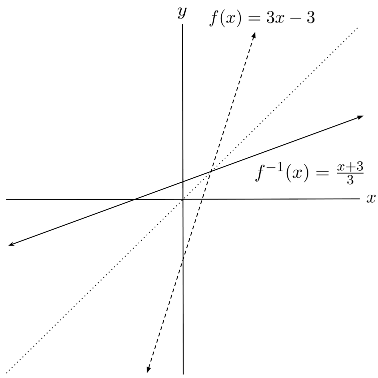


Chapter 7.4

1) $f^{-1}(x) = \frac{1}{2}x$

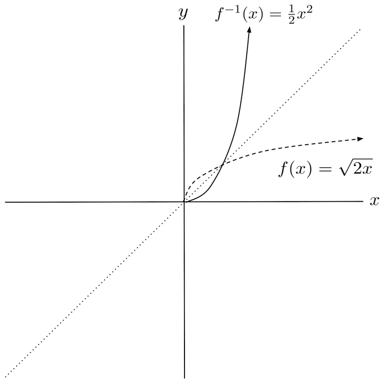


2) $f^{-1}(x) = \frac{x+3}{3}$



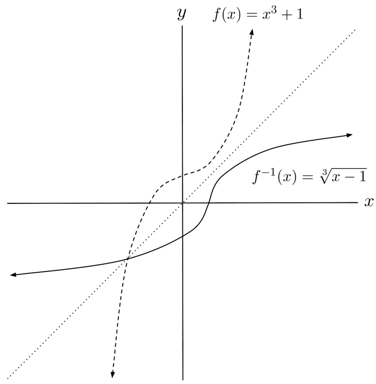
3) $f^{-1}(x) = \dfrac{1}{2}x^2$

restriction: $x \geq 0$



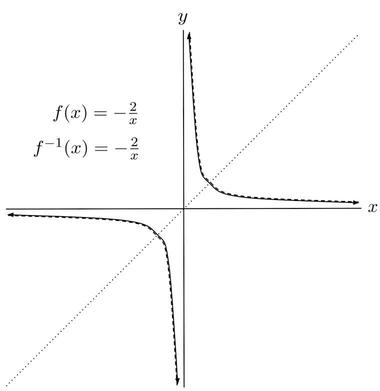
4) no inverse function

5) $f^{-1}(x) = \sqrt[3]{x-1}$

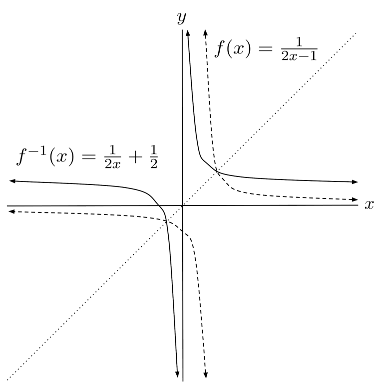


6) no inverse function

7) $f^{-1}(x) = -\frac{2}{x}$

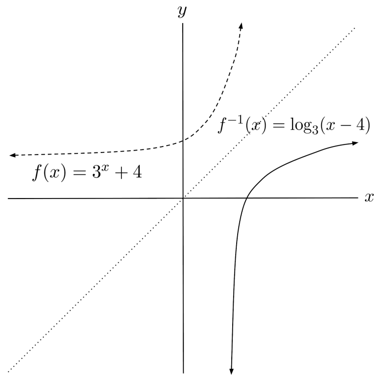


8) $f^{-1}(x) = \frac{1}{2x} + \frac{1}{2}$

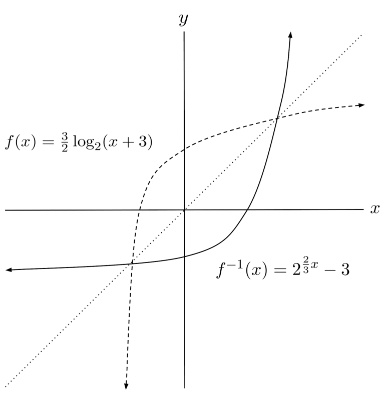


9) no inverse function

10) $f^{-1}(x) = \log_3(x-4)$

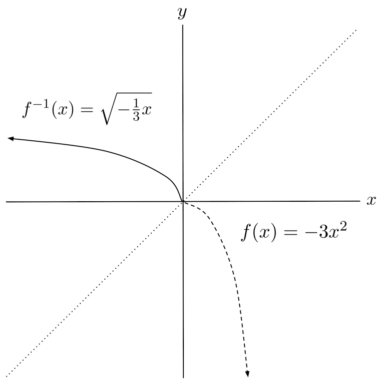


11) $f^{-1}(x) = 2^{\frac{2}{3}x} - 3$

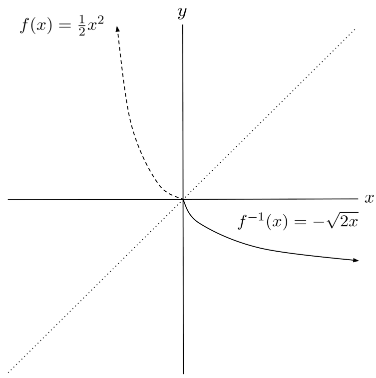


12) no inverse function

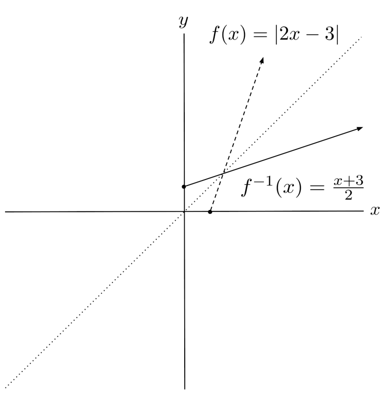
13) $f^{-1}(x) = \sqrt{-\frac{1}{3}x}$



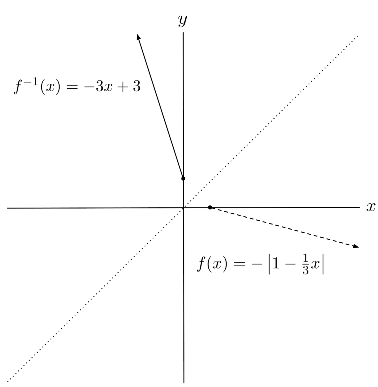
14) $f^{-1}(x) = -\sqrt{2x}$



15) $f^{-1}(x) = \frac{x+3}{2}$



16) $f^{-1}(x) = -3x+3$



Chapter 7.5

1) $2(x+5)^2$

2) $|4-x^5|$

3) $\sqrt{|-2^x+4|}$

4) $\sin\left(\frac{2x}{2x-1}\right)$

5) $\sqrt{1+\sqrt[3]{\sin^2 x}}$

6) $\left|\log\left(\tan\left(\sqrt{x}\right)\right)\right|^3$