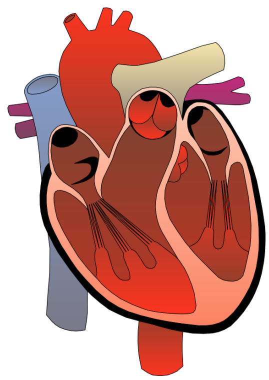
Connecting Calculus to the Real World

Justin Skycak, 2017

Calculus is much more fun to learn when we see how it connects to the real world. Here are some examples!

Science and Medicine

Cardiac Output



Calculus can be used to measure cardiac output, the volume of blood the heart pumps out into the aorta per unit time.

This is done via the dye dilution method, which involves injecting a known volume of dye into the blood, plotting the concentration of the dye as a function of time, and using the concentration function to figure out the blood's rate of flow.

To figure out the rate of flow, we set up an equation using the following variables:

A = the initial volume of dye
c*(t)* = the dye's concentration as a function of time
F = the volume of blood pumped out per unit time, i.e. the blood's rate of flow
T = the amount of time until the dye runs out

The amount of dye pumped out by the heart at any given point in time is calculated as ***F*** ****c****(t)*, the product of the volume of blood pumped out at that time and the concentration of dye in the blood at that time.

Once the dye runs out, the entire initial volume of dye should have been pumped out. Thus, A should equal the sum of ***F*** ****c****(t)* at all points in time, from 0 to ***T***. We calculate the sum via integration:

$$A = \int_0^T F * c(t)dt$$

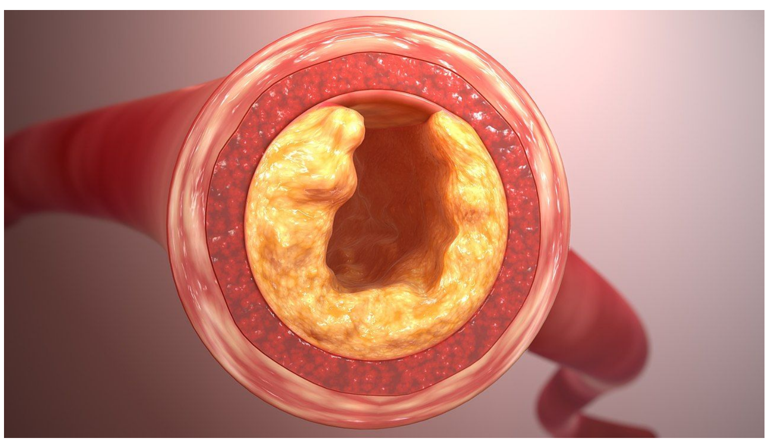
Because F is a constant, we can move it out of the integral:

$$A = F\int_0^T c(t)dt$$

Dividing, we reach a formula for ***F***:

$$F = \frac{A}{\int_0^T c(t)dt}$$

Understanding Plaque Buildup



Atherosclerosis occurs when fatty deposits, or plaque, builds up to clog arteries. Even seemingly small amounts of plaque can be problematic, because whenever the artery narrows by some percent, the amount of blood that can flow through it decreases by a much greater percent.

Calculus can help us understand how the rate of blood flow through an artery depends on its radius, and why even slight narrowing of arteries can pose such a big problem to blood flow.

Within an artery, blood moves at different speed. The blood near the center of the artery moves the fastest, while the blood near the walls of the artery moves the slowest due to friction against the walls.

When we measure this empirically, we find that if the artery has radius ***R***, then velocity of a blood flow at a radius ***r*** from the center is proportional to the difference of squares:

$$v(r) = k(R^2 - r^2)$$

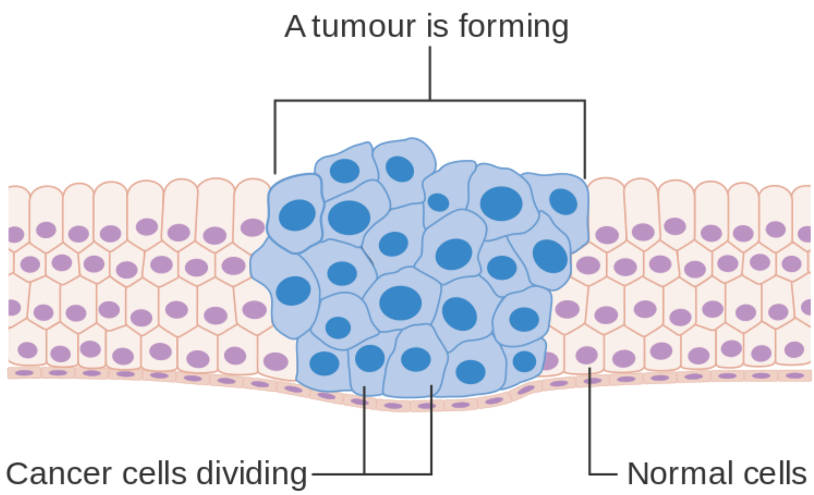
In order to find how much blood flows through a cross-section of the artery, we can integrate the velocity over the entire area of a cross section:

$$\begin{aligned} F &= \int v(r) dA \\ &= \int_0^R k(R^2 - r^2) * 2\pi r dr \\ &= 2\pi k \int_0^R R^2 r - r^3 dr \\ &= 2\pi k \left[\frac{R^2 r^2}{2} - \frac{r^4}{4} \right]_0^R \\ &= 2\pi k \left[\frac{R^4}{2} - \frac{R^4}{4} \right] \\ &= \pi k R^4 \end{aligned}$$

This formula is known as *Poiseuille's Law*, and it tells us that the flow through an artery is proportional to the fourth power of its radius.

This is why the narrowing of an artery can cause extreme decreases in blood flow. If an artery becomes halfway covered in plaque, then the blood flow becomes even less than half of what it used to be: only (0.5)^4 = 0.06 = 6% of the blood flow remains.

Modeling Tumor Growth



Calculus can help us model the growth of tumors.

Tumors appear to grow exponentially early in their lifecycles, which means that if a tumor's volume at time ***t*** is given by ***V(t)*** and its growth rate is a constant ***r***, then

$$\frac{dV}{dt} = rV$$

However, tumors do not grow exponentially forever: it has been observed that after some time, tumor growth slows down.

To incorporate this into our model, instead of setting the growth rate to a constant ***r***, we can set it to an exponential decay function given by

$$r(t) = r_0 e^{-kt}$$

Then the full model is

$$\frac{dV}{dt} = r_0 e^{-kt} V$$

We can separate variables and integrate to solve for ***V***:

$$
\begin{aligned}
\frac{dV}{dt} &= r_0 e^{-kt} V \\
\frac{dV}{V} &= r_0 e^{-kt} dt \\
\int \frac{dV}{V} &= \int r_0 e^{-kt} dt \\
\ln V &= \text{constant} + \int r_0 e^{-kt} dt \\
V &= e^{\text{constant} + \int r_0 e^{-kt} dt} \\
V &= (\text{constant}) e^{\int r_0 e^{-kt} dt} \\
V &= (\text{constant}) e^{\text{constant} - \frac{r_0}{k} e^{-kt}} \\
V &= C_1 e^{C_2 - \frac{r_0}{k} e^{-kt}}
\end{aligned}
$$

We can solve for one of the constants in terms of the initial volume and the other constant:

$$
\begin{aligned}
V_0 &= V(t = 0) \\
V_0 &= C_1 e^{C_2 - \frac{r_0}{k}} \\
C_1 &= V_0 e^{\frac{r_0}{k} - C_2}
\end{aligned}
$$

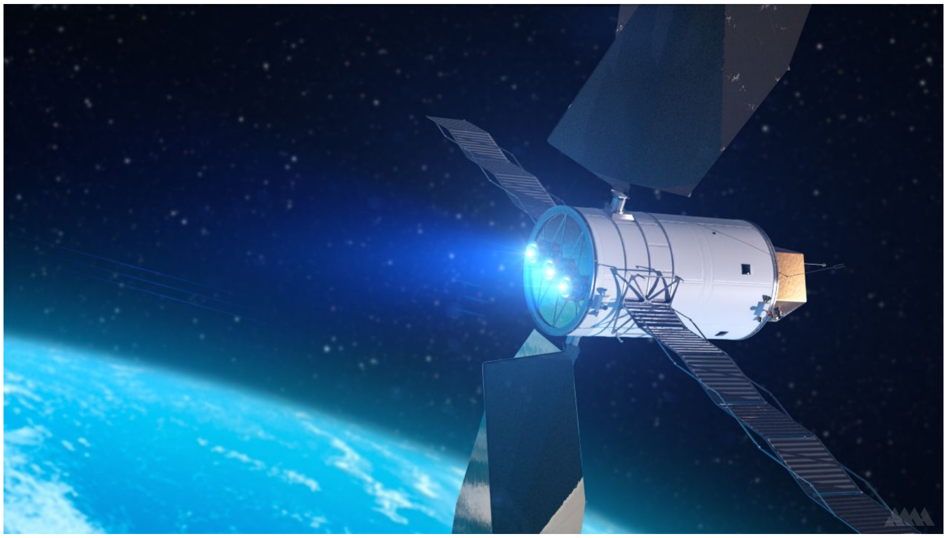
When we plug the constant back in, it cancels out the other constant to yield a final formula:

$$
\begin{aligned}
V &= \left(V_0 e^{\frac{r_0}{k} - C_2} \right) e^{C_2 - \frac{r_0}{k} e^{-kt}} \\
&= V_0 e^{\frac{r_0}{k} - C_2 + C_2 - \frac{r_0}{k} e^{-kt}} \\
&= V_0 e^{\frac{r_0}{k} \left(1 - e^{-kt} \right)}
\end{aligned}
$$

This is called the Gompertz function, and it has been used to model tumor growth and measure the effectiveness of tumor-killing treatments.

Technology and Engineering

Rocket Propulsion



What speed can a rocket reach if it has a particular amount of fuel? This is a difficult question because when a rocket burns some amount of fuel, it gets lighter, so burning the same amount of fuel again will speed the rocket up even more. But calculus can help us answer it.

Let's think of a rocket burning a small bit of fuel and speeding up by a small bit. We have the following variables:

Δm = mass of small bit of fuel to be burnt
m = mass of fueled rocket after burning the bit of fuel
v = velocity of fueled rocket before burning the bit of fuel
Δv = small increase in velocity due to burning the small bit of fuel
u = velocity of fuel exhaust, from the rocket's frame of reference

The law of conservation of momentum tells us that the momentum (mass times velocity) of the rocket and fuel should stay the same even after a bit of fuel is burnt.

The momentum before the fuel is burnt is obtained by multiplying the mass of the fueled rocket (including the small bit of fuel that hasn't been burnt yet) with the velocity of the fueled rocket before burning the fuel. So, it is

$$(m + \Delta m)v$$

The momentum of the rocket after the fuel is burnt is obtained by multiplying the mass of the fueled rocket (excluding the small bit of fuel that was burnt) with the slightly increased velocity of the fueled rocket after burning the fuel. So, it is

$$m(v + \Delta v)$$

However, the burnt bit of fuel also shoots away from the rocket as exhaust, giving additional momentum

$$\Delta m(v - u)$$

Consequently, the total momentum after burning fuel is

$$m(v + \Delta v) + \Delta m(v - u)$$

Setting momentum before fuel-burning equal to momentum after fuel-burning, we have

$$\begin{aligned}(m + \Delta m)v &= m(v + \Delta v) + \Delta m(v - u) \\ mv + \Delta mv &= mv + m\Delta v + \Delta mv = \Delta mu \\ m\Delta v &= u\Delta m\end{aligned}$$

In the limit as the amount of burnt fuel gets infinitely small, we can replace the deltas with differentials. Since ejecting a positive ***Δm*** results in a negative change in mass for the rocket, we use ***Δm*** = ***- dm***.

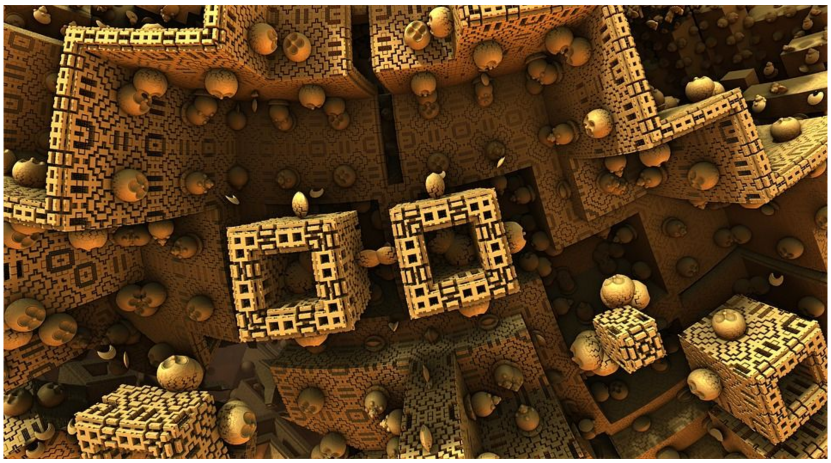
$$mdv = -udm$$

Finally, we separate variables and solve the differential equation. (Keep in mind that ***u*** is constant.)

$$\begin{aligned}\int dv &= -u \int \frac{dm}{m} \\ v_{final} - v_{initial} &= -u \left[\ln(m)\right]_{m_{initial}}^{m_{final}} \\ v_{final} &= v_{initial} - u\left[\ln(m_{final}) - \ln(m_{initial})\right] \\ v_{final} &= v_{initial} + u\left[\ln(m_{initial}) - \ln(m_{final})\right] \\ v_{final} &= v_{initial} + u\ln\left(\frac{m_{initial}}{m_{final}}\right)\end{aligned}$$

This is called the *ideal rocket equation*, which can be applied to orbital maneuvers in order to determine how much fuel is needed to change to a particular new orbit, or to find the new orbit as the result of burning some amount of fuel.

Rendering 3D Computer Graphics



By including lighting and shadows, 2D computer graphics can be made to appear 3D. The process by which this is done is called *rendering*, and it makes use of calculus to simulate the light that bounces around in a scene as it makes its way to the camera.

Suppose we are looking at point $\boldsymbol{x}$ from angle $\boldsymbol{\theta_0}$. We see light coming sources (e.g. sun, overhead lights) which we will call $\boldsymbol{L_{direct}(x, \theta_0)}$, and there is also light that hits the point from other angles $\boldsymbol{\theta}$ after bouncing around. We will call this bounced light $\boldsymbol{L_{bounce}(x, \theta)}$. However, only a proportion of bounced light that hits the point will reflect to our viewing angle $\boldsymbol{\theta_0}$. We will call this proportion $\boldsymbol{p(x, \theta, \theta_0)}$.

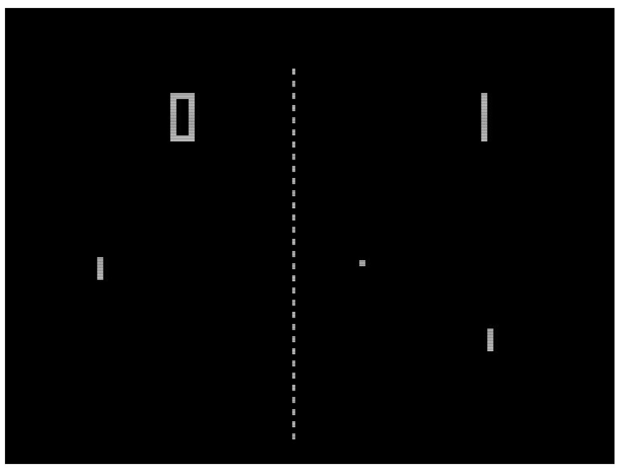
The total light coming to us when we look at point $\boldsymbol{x}$ from angle $\boldsymbol{\theta_0}$ is calculated by adding up all the light arriving from sources (like the sun, or overhead lights) as well as the light reflected after bouncing off other surfaces in the scene. To sum up all the light reflected to us after bouncing off other surfaces, we have to integrate over the hemisphere.

Thus, we calculate the total light we see at point ***x*** from angle $\boldsymbol{\theta_0}$ using the following formula:

$$L_{total}(x, \theta_0) = L_{direct}(x, \theta_0) + \int p(x, \theta, \theta_0) L_{bounce}(x, \theta) \cos(\theta) d\theta$$

This integral is called *Kajiya's Rendering Equation*, and it is often approximated in computer graphics to compute the total light coming from a point when viewed from a particular angle.

Physics Engines in Video Games



Physics engines allow video game characters to move in realistic virtual environments, so that if your character throws or jumps or drives a car off a cliff, the result will mimic reality. To perform this task, the physics engine periodically updates the locations of objects, such that the trajectories of objects follow physical laws.

For early games like pong, the physical laws were simple. In pong, the ball travels with a constant velocity ***v*** at an angle $\boldsymbol{\theta}$, and it deflects off paddles such that its incoming angle is equal to its outgoing angle.

The velocity affects the ball's ***x*** and ***y*** positions as follows:

$$\frac{dx}{dt} = v\cos(\theta)\,\frac{dx}{dt} = v\cos(\theta)$$
$$\frac{dy}{dt} = v\sin(\theta)\,\frac{dy}{dt} = v\sin(\theta)$$

Using differentials, we write this as

$$dx = v\cos(\theta)dt$$
$$dy = v\sin(\theta)dt$$

The updates for a ball's motion across the grid, then, are

$$x \to x + (v\cos\theta)\Delta t$$
$$y \to y + (v\sin\theta)\Delta t$$

where **Δt** is the timestep (the fraction of a second to be simulated) and ***θ*** is updated according to

$$\theta \to 180^\circ - \theta$$

whenever the ball hits a surface.

In more complex modern games, however, game objects are assigned properties like mass and friction constants, so that their trajectories can be calculated in response to physical forces. This is made possible through Newton's second law of motion, which describes how a force ***F*** affects an object's velocity ***v***.

$$F = m\frac{dv}{dt}$$

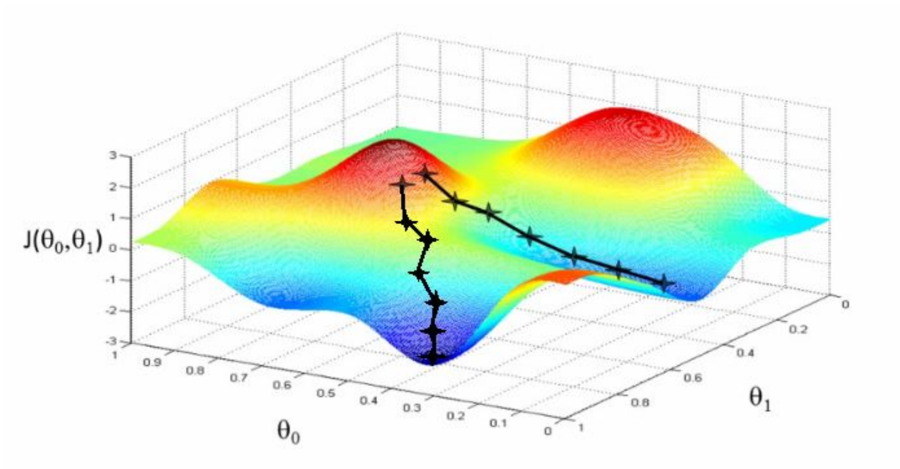
In 1 dimension, the object's velocity ***v*** then affects its position ***x*** according to

$$v = \frac{dx}{dt}$$

Thus, the update equations for an object experiencing a force in 1 dimension are given by

$$x \to x + v\Delta t$$
$$v \to v + \frac{F}{m}\Delta t$$

Optimization via Gradient Descent



In technology and engineering, we are often faced with the task to optimize some function which represents a system's performance as a function of the parameters of its components. However, the function to be optimized is often so complicated that you can't solve it with pencil and paper, and it may even contain too many variables to graph on your computer.

Fortunately, the concept of the derivative makes it possible to optimize many of these equations.

For multivariable functions, the derivative is called the gradient. Optimization problems are normally stated as finding the parameters which minimize a function, and we can use the gradient to "descend" down the landscape of the function into its lowest points. (if you want to maximize instead, you can turn it into a minimization problem by tacking on a negative.)

The method of gradient descent starts with an initial guess at the parameters, which places you somewhere on a "mountain" in the function's graph. Your goal is to get down from the mountain, into a valley, but there is a heavy fog that makes it impossible to see around you. Since you can't see where the valley is, you have to figure out how to guess the correct direction to go by looking only at the ground you stand on.

The key is to pay attention to the gradient, or steepness. If you take a step forward and you notice that your forward foot lands higher, then you're probably going up the mountain, so you shouldn't continue in that direction. If you take a step forward and your forward foot lands lower, then you're probably going down the mountain, so you should continue. And if you keep on taking steps down the mountain, you'll eventually reach the bottom.

This is the method of gradient descent: descend down the path where the gradient is the steepest.

Business and Economics

Maximizing Profit

Businesses wish to maximize their profit, the amount of money they get to keep once they've sold products and paid off the cost of producing those products. To calculate product, you start with the total revenue (the total amount of money the business makes from selling products), and then subtract the total costs (the total amount of money the business spends on producing products).

The total revenue and total cost will depend on how many units of product a business produces. If they sell too few units, they may be able to command a high price for their product, but the high price will be offset by a high cost of producing those few units. On the other hand, if they produce too many units, scaling up their production process and making it more efficient may lead to lower costs per unit, but low costs will be offset by an even lower market price. To maximize profit, a business wants to produce just the right number of units.

If we know what the revenue and costs are of producing any number x of units, then we can use calculus to figure out what number of units to produce.

Let ***R(x)*** be the total revenue of producing ***x*** units, and ***C(x)*** be the total cost of producing ***x*** units. Then the profit ***P(x)*** of producing ***x*** units is given by

$$P(x) = R(x) - C(x)$$

We can find the maximum profit by taking the derivative and setting it equal to zero:

$$\begin{aligned} 0 &= P'(x) \\ 0 &= R'(x) - C'(x) \\ C'(x) &= R'(x) \end{aligned}$$

C'(x) is the marginal cost, the cost of producing another unit, and ***R'(x)*** is the marginal revenue, the money we make from selling another unit. Thus, the maximum profit occurs when marginal cost equals marginal revenue, when the cost of producing another unit equals the money we can make by selling another unit. Intuitively, that is the break-even point: if we sell another unit for the same amount of money it takes to produce it, the total profit is unaffected.

Continuously Compounded Interest

Suppose you invest a principal amount of money, ***P***, into an account with interest rate ***r*** which is compounded ***n*** times per year. After ***t*** years, there would be ***nt*** total compoundings, each by a factor ***1+(r/n)***, so your principal amount grows to an amount

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Annual compounding means ***n=1***. Monthly compounding means ***n=12***. Daily compounding means ***n=365***. If you compound every minute, then ***n=(365)(24)(60)=525600***. If you compound continuously, then you take the limit as ***n*** goes to infinity.

You might recall "Pert," the formula for a continuously compounded investment:

$$A = Pe^{rt}$$

But where does that formula come from?

We can use calculus to derive it.

Taking the limit as ***n*** approaches infinity, we have

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = P\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right]^{t}$$

To derive the formula, we need to show that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n} = e^{r}$$

By taking the natural log and rearranging the expression, we can get something that looks a bit like a difference quotient:

$$\begin{aligned}
\ln\left[\left(1+\frac{r}{n}\right)^{n}\right] &= n\ln\left(1+\frac{r}{n}\right) \\
&= \frac{r\ln\left(1+\frac{r}{n}\right)}{r/n} \\
&= r\left[\frac{\ln\left(1+\frac{r}{n}\right)-0}{r/n}\right] \\
&= r\left[\frac{\ln\left(1+\frac{r}{n}\right)-\ln(1)}{r/n}\right]
\end{aligned}$$

Define ***h=r/n***. As ***n*** goes to infinity, ***h*** goes to 0, so

$$\begin{aligned}\lim_{n\to\infty} \ln\left[\left(1+\frac{r}{n}\right)^n\right] &= \lim_{n\to\infty} r\left[\frac{\ln\left(1+\frac{r}{n}\right)-\ln(1)}{r/n}\right] \\ &= \lim_{h\to 0} r\left[\frac{\ln(1+h)-\ln(1)}{h}\right] \\ &= r\lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}\end{aligned}$$

Now it looks exactly like a difference quotient. In fact, it is the derivative of the natural log taken at 1:

$$\begin{aligned}\lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} &= \frac{d}{dx}\left[\ln(x)\right]_{x=1} \\ &= \left[\frac{1}{x}\right]_{x=1} \\ &= 1\end{aligned}$$

So,

$$\begin{aligned}\lim_{n\to\infty} \ln\left[\left(1+\frac{r}{n}\right)^n\right] &= r\lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} \\ &= r(1) \\ &= r\end{aligned}$$

Since order does not matter with logs and limits (the log of a limit is the limit of the log), we also have

$$\begin{aligned}r &= \lim_{n\to\infty} \ln\left[\left(1+\frac{r}{n}\right)^n\right] \\ &= \ln\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n\right].\end{aligned}$$

Using the definition of natural logarithm,

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r.$$

Finally,

$$\begin{aligned}
\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} &= P\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right]^{t} \\
&= P\left[e^{r}\right]^{t} \\
&= Pe^{rt}
\end{aligned}$$

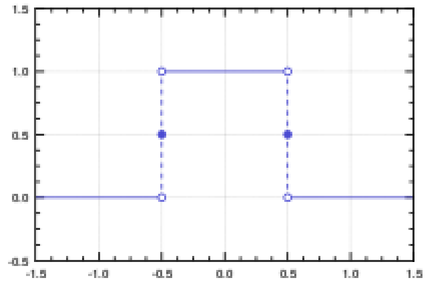
History and Philosophy

The Man who “Broke” Math

Taylor series allow one to write differentiable functions as infinite polynomials. Fourier series are similar, except instead of an infinite polynomial, they use the sum of infinitely many frequencies of sine and cosine waves. Fourier series have seen many applications across mathematics, physics, and engineering, but when Joseph Fourier first introduced them in the beginning of the 19th century, they gave mathematicians nightmares.

Sine and cosine are both continuous functions, so when you add a bunch of them together, the result is continuous. At the time mathematicians had not given much thought to the difference between finite sums and infinite sums, so it was assumed that if you add together infinitely many continuous functions like sine and cosine waves, the result would still be continuous.

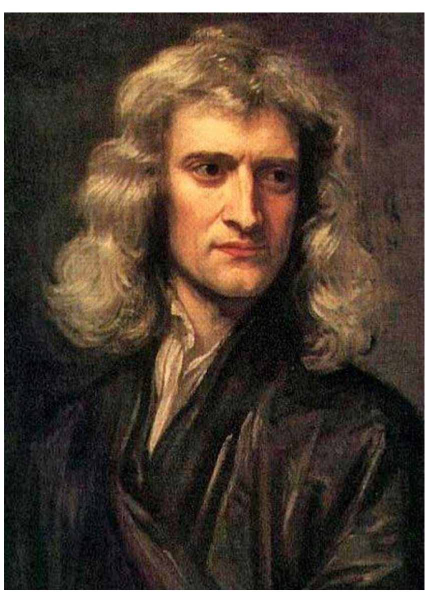
However, this turned out to be incorrect. There is a Fourier series which corresponds to the discontinuous step function shown below.



This was troubling to the mathematical community, because all work based on infinite sums was suddenly on shaky ground. Consequently, during the rest of the 19th century, there was a movement in the mathematical community to reformulate previous findings in a more rigorous fashion. This way, they could be certain that they were correct, and there would be no worrisome “surprises” like Fourier’s discontinuous series in the future.

To this end, Augustin-Louis Cauchy, Bernhard Riemann, and Karl Weierstrass reformulated calculus in what is now known as real analysis. Later on in the 19th century, Georg Cantor established the first foundations of set theory, which enabled the rigorous treatment the notion of infinity. Since then, set theory has become the common language of nearly all of mathematics.

The Newton-Leibniz Controversy




The Newton-Leibniz Controversy was a dispute between the 17th-century mathematicians Isaac Newton and Gottfried Leibniz, over who had first invented calculus.

Newton had begun working on his form of calculus in during his spare time in 1666, when he and many other students were sent home from Cambridge on account of the plague. He developed his method in order to solve physics problems, and he called it the “method of fluxions” (“fluxion” was his term for the derivative). However, he kept his findings to himself for nearly two decades, until publishing his *Philosophae Naturalis Mathematica* in 1687, which presented new (and now essential) theories of physical motion and used calculus to back them up.

Leibniz, on the other hand, only waited a decade to publish his results after finding the area under the curve in 1675. He published an explanation of differential calculus in 1684, and of integral calculus in 1686.

Since Leibniz published first, he received sole credit for discovering calculus. But Newton believed that Leibniz had plagiarized his own work, and had some circumstantial evidence in his favor: their network of colleagues overlapped, so Leibniz may have seen unpublished manuscriopts, and some of Newton’s first descriptions of calculus (his “method of fluxions”) had appeared in a letter of correspondence to Leibniz.

Newton set out to prove this, and since his *Philosophae* had gained sufficient popularity to make him a scientific celebrity, he won the support of the British Royal Society. Leibniz, having few allies, was defenseless, and lived just long enough to hear the Royal Society proclaim Newton the sole discoverer of calculus a year before his death in 1715.

But nobody came out of the dispute well.

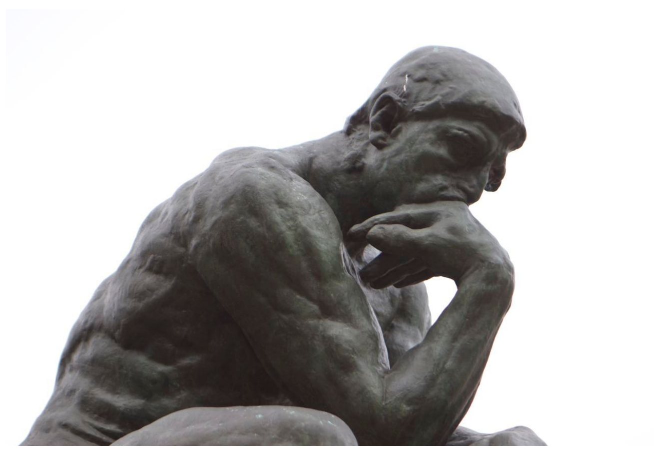
Leibniz had formulated calculus using a notation vastly more efficient that Newton’s, but because Newton claimed that the notation was meant to hide Leibniz’s treachery, England refused to use Leibniz’s notations as a matter of national pride (Leibniz was German). As a result of using outdated notation, British math fell behind the rest of the continent.

After reviewing Newton and Leibniz’s papers and correspondences, most modern historians have concluded that, although Leibniz likely saw some early manuscripts of Newton’s *Philosophae*, he had already come to his own conclusions by that time. Whereas Newton’s grounded his calculus in limits and concrete physical reality, Leibniz’s work was inspired by more abstract concepts and theory. Newton and Leibniz had invented calculus independently.

But even to say that Newton and Leibniz were the “sole inventors” of calculus, would be to discredit centuries of mathematical ideas leading up to the official “discoveries.” In the times of ancient Greece, Archimedes was the first to find the tangent line to a curve, and Antiphon developed the method of exhaustion for calculating area. In India, centuries before the European Enlightenment, Aryabhata expressed an astronomical problem as a differential equation, and Parameshvara developed an early version of the mean value theorem. Even during the Enlightenment, Fermat, Pascal, and Barrow had developed the concept of the derivative. Barrow even offered the first proof of the fundamental theorem of calculus, which links differentiation with integration. And Newton was Barrow's own student.

Though calculus may often be said to be the creation of Newton and Leibniz, it was truly an invention of many mathematicians, over many centuries.

A Failure of Intuition



Calculus can show us how our intuition can fail us, a common theme in philosophy. When we question our intuitions and analyze a problem in depth, we sometimes come up with wildly different answers than we had expected. Our confusion can lead us to think about the problem in different ways, and build a new intuition which will actually lead us to the right answer.

We'll demonstrate this process on a limit problem.

At first glance, the limit

$$\lim_{x \to \infty} \sqrt{x^2 + x} - \sqrt{x^2}$$

looks like something we should be able to tackle with intuition.

If we remove the square roots, the limit comes out to infinity:

$$\begin{aligned}\lim_{x\to\infty}(x^2+x)-x^2 &= \lim_{x\to\infty} x\\ &= \infty\end{aligned}$$

The square root of infinity is infinity, so shouldn't the original limit come out to infinity?

It turns out, this is not the case. Here is what happens if we use formal reasoning to figure the limit out step-by-step:

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^2+x}-\sqrt{x^2} &= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+x}-\sqrt{x^2}\right)\left(\sqrt{x^2+x}+\sqrt{x^2}\right)}{\sqrt{x^2+x}-\sqrt{x^2}}\\
&= \lim_{x\to\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}-\sqrt{x^2}}\\
&= \lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}-\sqrt{x^2}}\\
&= \lim_{x\to\infty}\frac{x\left(\frac{1}{x}\right)}{\left(\sqrt{x^2+x}-\sqrt{x^2}\right)\left(\frac{1}{x}\right)}\\
&= \lim_{x\to\infty}\frac{1}{\frac{\sqrt{x^2+x}}{x}-\frac{\sqrt{x^2}}{x}}\\
&= \lim_{x\to\infty}\frac{1}{\frac{\sqrt{x^2+x}}{\sqrt{x^2}}-\frac{\sqrt{x^2}}{\sqrt{x^2}}}\\
&= \lim_{x\to\infty}\frac{1}{\sqrt{\frac{x^2+x}{x^2}}-1}\\
&= \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}-1}\\
&= \frac{1}{\sqrt{1+0}-1}\\
&= \frac{1}{2}\end{aligned}$$

Where does that number come from? It's hard to see in the original limit.

However, there is intuition which can motivate the correct answer. If we allow ourselves to add a constant term inside the limit, then it all makes sense:

$$
\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+x} - \sqrt{x^2} &= \lim_{x\to\infty} \sqrt{x^2+x+\frac{1}{4}} - \sqrt{x^2} \\
&= \lim_{x\to\infty} \sqrt{\left(x+\frac{1}{2}\right)^2} - \sqrt{x^2} \\
&= \lim_{x\to\infty} \left(x+\frac{1}{2}\right) - x \\
&= \lim_{x\to\infty} \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

Now that we've rebuilt our intuition about these kinds of problems, we can tackle similar ones by adding constants within square roots. For example, using our new intuition, we find that

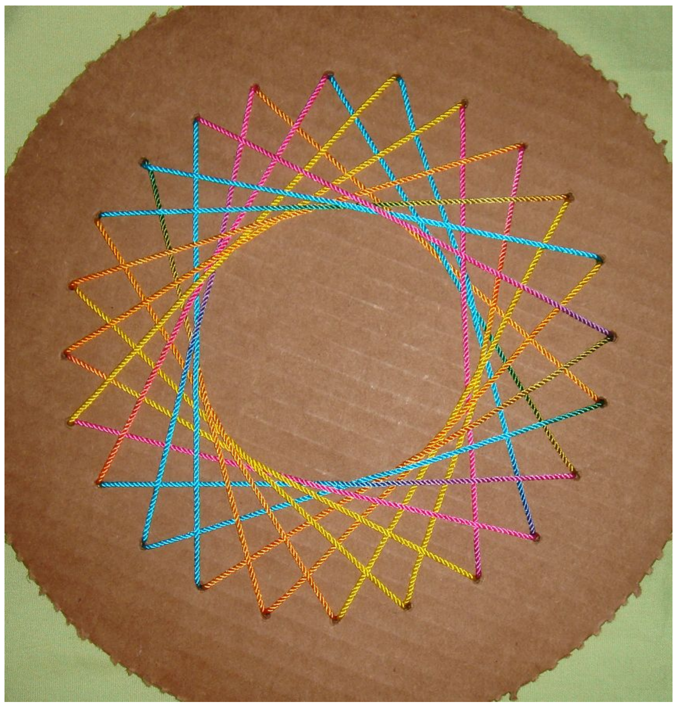
$$
\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+2x} - \sqrt{x^2+5} &= \lim_{x\to\infty} \sqrt{x^2+2x+1} - \sqrt{x^2} \\
&= \lim_{x\to\infty} \sqrt{(x+1)^2} - \sqrt{x^2} \\
&= \lim_{x\to\infty} (x+1) - x \\
&= \lim_{x\to\infty} 1 \\
&= 1
\end{aligned}
$$

Working the problem out the long way, we see that our new intuition is correct!

$$
\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+2x}-\sqrt{x^2+5} &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x}-\sqrt{x^2+5}\right)\left(\sqrt{x^2+2x}+\sqrt{x^2+5}\right)}{\sqrt{x^2+2x}+\sqrt{x^2+5}} \\
&= \lim_{x\to\infty} \frac{(x^2+2x)-(x^2+5)}{\sqrt{x^2+2x}+\sqrt{x^2+5}} \\
&= \lim_{x\to\infty} \frac{2x-5}{\sqrt{x^2+2x}+\sqrt{x^2+5}} \\
&= \lim_{x\to\infty} \frac{(2x-5)\left(\frac{1}{x}\right)}{\left(\sqrt{x^2+2x}+\sqrt{x^2+5}\right)\left(\frac{1}{x}\right)} \\
&= \lim_{x\to\infty} \frac{2+\frac{5}{x}}{\frac{\sqrt{x^2+2x}}{\sqrt{x^2}}+\frac{\sqrt{x^2+5}}{\sqrt{x^2}}} \\
&= \lim_{x\to\infty} \frac{2+\frac{5}{x}}{\sqrt{\frac{x^2+2x}{x^2}}+\sqrt{\frac{x^2+5}{x^2}}} \\
&= \lim_{x\to\infty} \frac{2+\frac{5}{x}}{\sqrt{1+\frac{2}{x}}+\sqrt{1+\frac{5}{x^2}}} \\
&= \frac{2}{\sqrt{1}+\sqrt{1}} \\
&= 1
\end{aligned}
$$

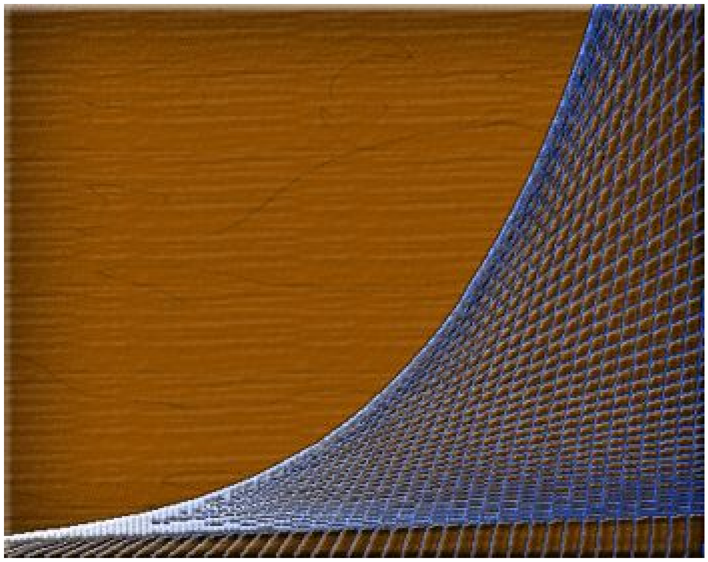
Arts and Athletics

Derivatives in String Art

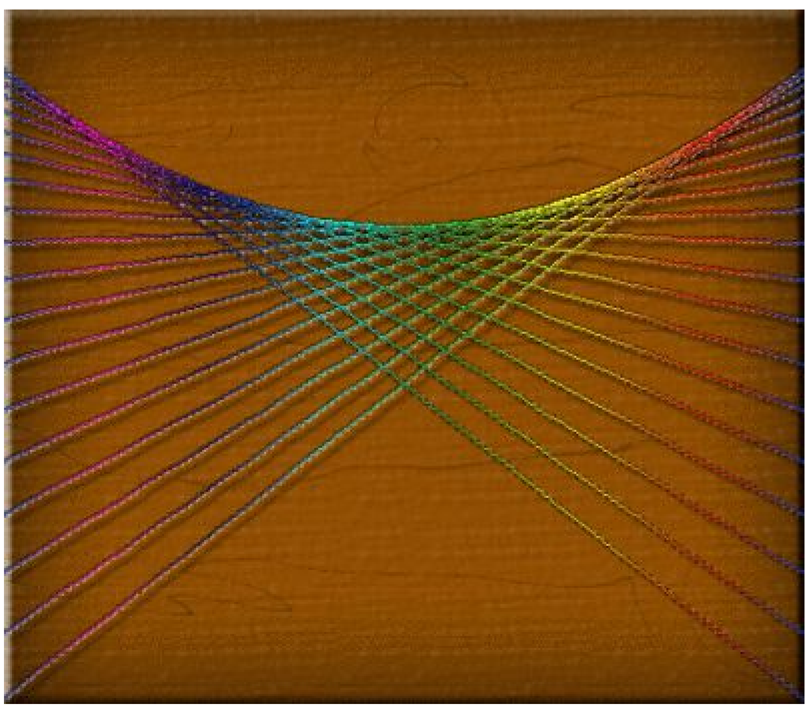


String art involves creating curves, using only straight lines. The reason it works is because the lines “slope” into each other, becoming tangent lines to the curve. For example, the lines above are the tangent lines of the circle, where the slope of the line matches derivative of the circle at each point.

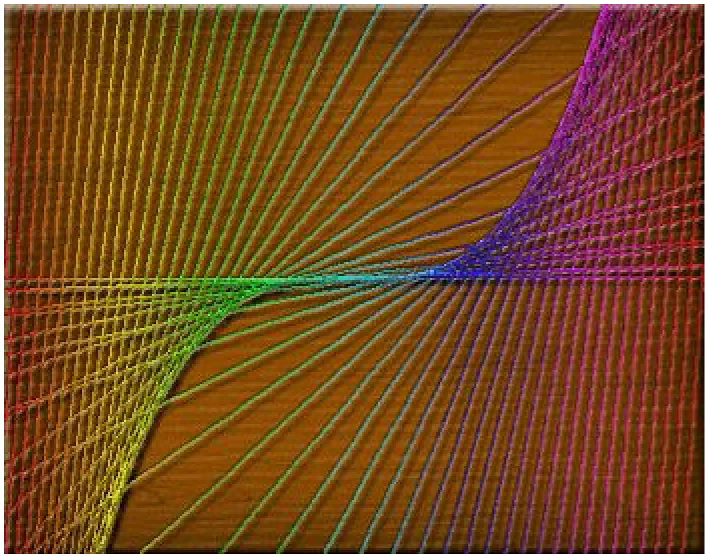
Next, tangent lines to an exponential function:



A quadratic function:



A cubic:



And finally, a sine wave:

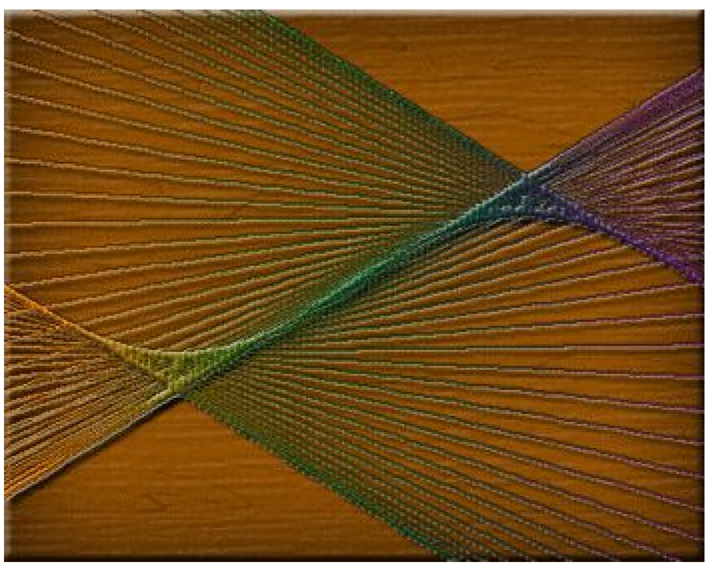


Image credits: Math Images - The Math Forum @ Drexel
http://mathforum.org/mathimages/index.php/String_Art_Calculus

Calculating the Horsepower of an Offensive Lineman

Hearing the word “horsepower,” many people think of car engines. Horsepower measures the rate of work, or energy expenditure, so it is commonly used as a performance metric to compare engines.

However, horsepower is not a familiar metric like height or weight. We know the heights and weights of everyday objects, so it is easy for us to know that 1000 pounds is very heavy to lift, and a person who is 4 feet tall is considered short. But when we say that a pickup truck has 300-400 horsepower, what can we compare it to?

Do you know the horsepower of a human, say, an NFL offensive lineman?

Let’s find out.

An offensive lineman’s power can be calculated as the product of his force and maximum velocity:

$$P = Fv_{max}$$

And force is given by the product of his mass and his acceleration:

$$F = ma$$

Mass is easy to find: a typical lineman weighs about 320 pounds, or 145 kilograms. Acceleration and maximum velocity are more difficult to find, but using calculus, we can find them from a lineman’s 40-yard dash time.

Say the lineman runs a 40-yard dash in 5.3 seconds, reaching top speed 2 seconds after starting. 40 yards is 36.6 meters, so we have

$$36.6 = (\text{distance while accelerating}) + (\text{distance at top speed})$$

First, we need to use calculus to solve for the distance while accelerating. Since acceleration is the derivative of velocity, velocity is the integral of acceleration:

$$a = \frac{dv}{dt} \Rightarrow v(t) = \int adt$$

Therefore, the velocity is given by

$$v(t) = at + C$$

where C is a constant. But since the velocity at time t = 0 is 0, we have

$$\begin{aligned} 0 &= v(t = 0) \\ &= a(0) + C \\ &= C \end{aligned}$$

and hence the velocity is given by

$$v(t) = at$$

Likewise, since velocity is the derivative of distance, distance is the integral of velocity:

$$v(t) = \frac{dx}{dt} \Rightarrow x(t) = \int v(t)dt$$

Therefore, the distance is given by

$$\begin{aligned} x(t) &= \int v(t)dt \\ &= \int atdt \\ &= \frac{a}{2}t^2 + C \end{aligned}$$

where C is again a constant.

Again, since the distance at time t = 0 is 0, we have

$$\begin{aligned} 0 &= x(t=0) \\ &= \frac{a}{2}(0)^2 + C \\ &= C \end{aligned}$$

and hence the distance is given by

$$x(t) = \frac{a}{2}t^2$$

Since the lineman stops accelerating at 2 seconds, the distance covered during this period of acceleration is given by

$$\begin{aligned} x(2) &= \frac{a}{2}(2)^2 \\ &= 2a \end{aligned}$$

Now, we need to find the distance at top speed. The lineman is at top speed for 5.3 - 2 = 3.3 seconds, and is traveling at a top speed of

$$\begin{aligned} v_{max} &= v(2) \\ &= 2a \end{aligned}$$

so the distance covered is given by

$$\begin{aligned} \text{distance at top speed} &= v_{max}(3.3) \\ &= (2a)(3.3) \\ &= 6.6a \end{aligned}$$

Substituting back into our original equation, we have

$$\begin{aligned} 36.6 &= (\text{distance while accelerating}) + (\text{distance at top speed}) \\ 36.6 &= 2a + 6.6a \\ 36.6 &= 8.6a \\ 4.6 &= a \end{aligned}$$

The lineman accelerates at 4.6 m/s^2.

Also, solving for maximum velocity:

$$\begin{aligned} v_{max} &= 2a \\ &= 2(4.6) \\ &= 9.2 \end{aligned}$$

The lineman's maximum speed is at 9.2 m/s.

Plugging in to find the lineman's power, we have

$$\begin{aligned} P &= Fv_{max} \\ &= mav_{max} \\ &= (145)(4.6)(9.2) \\ &= 6136 \end{aligned}$$

This measurement is in watts. Converting to horsepower, we divide by the conversion factor 745.7 W = 1hp, reaching 8.2 horsepower, roughly a fortieth of that of a truck.